

The Einstein-Friedrich-nonlinear scalar field system and the stability of scalar field Cosmologies

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Abstract

A frame representation is used to derive a first order quasi-linear symmetric hyperbolic system for a scalar field minimally coupled to gravity. This procedure is inspired by similar evolution equations introduced by Friedrich to study the Einstein-Euler system. The resulting evolution system is used to show that, for some classes of scalar field potentials, small nonlinear perturbations of an expanding Friedmann-Robertson-Walker background with a selfinteracting scalar field source, decay exponentially to zero or converge to constant values. As an application of our results we discuss the nonlinear stability of flat power-law solutions produced by an exponential potential. It is found that for a certain range of the parameter p , there is stability even in the absence of accelerated expansion.

1 Introduction

An important problem of classical mathematical Cosmology concerns the asymptotic stability of spatially homogeneous and isotropic spacetimes. Within this class of spacetimes, those having a nonlinear scalar field as the matter model have been extensively used to model early and late times cosmological scenarios —see e.g. [1, 2, 3] and references therein. Recently, Ringström [11, 12] has proved that small perturbations of the initial data of scalar field cosmological solutions to the *Einstein Field Equations* (EFE) with accelerated expansion have maximal globally hyperbolic developments that are future causally geodesically complete. In particular, in [12] the case of power-law solutions due to an exponential potential has been considered as an application of the more general methods developed in [11]. The nonlinear asymptotic stability of power-law inflation solutions has also been discussed by Heinzle & Rendall [13] using Kaluza-Klein reductions and the methods of Anderson [17] —the latter, in turn, inspired by Friedrich’s analysis of the stability of the de Sitter spacetime [4].

A natural way to analyse the stability of spacetimes is to ask whether small perturbations of a given solution to the EFE asymptotically decay to a background solution. Most of approaches to this question have been limited to the use of linear or higher-order truncated perturbation theory, and thus, they never take fully into account the nonlinearity of the EFE —see e.g. [14, 15, 16] and also [22]. This type of analysis has been hampered by the lack of a suitable formulation of the EFE for which the theory of systems of first order hyperbolic partial differential equations can be applied.

In [5] Friedrich has introduced a frame representation of the vacuum EFE. The evolution equations implied by this alternative representation of the equations of General Relativity constitute a *first-order quasi-linear symmetric hyperbolic system (FOSH)*. In general, these systems are of the form

$$\mathbf{A}^0(t, \mathbf{x}, \mathbf{u})\partial_t \mathbf{u} - \mathbf{A}^j(t, \mathbf{x}, \mathbf{u})\partial_j \mathbf{u} = \mathbf{B}(t, \mathbf{x}, \mathbf{u})\mathbf{u} \quad (1)$$

where $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ is a smooth vector-valued function of dimension s with domain in $\Sigma \times [0, T]$ with Σ a spacelike 3-dimensional manifold. Moreover, \mathbf{A}^0 , \mathbf{A}^j , $j = 1, 2, 3$, and \mathbf{B} denote smooth $s \times s$ matrix valued-functions. The matrices \mathbf{A}^0 and \mathbf{A}^j are symmetric. In addition, the matrix \mathbf{A}^0 is positive definite.

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The operators ∂_t and ∂_j stand, respectively, for the partial derivatives with respect to the coordinates t and x^j .

The construction for vacuum spacetimes given in [5] has been extended in [6, 7] to the case of a *perfect fluid* using a Lagrangian description of the fluid flow —see also [24, 25]. In both the vacuum and the perfect fluid cases the introduction of a frame formalism gives rise to extra gauge freedom. This freedom is associated to the evolution of the spatial frame coefficients along the flow of the time-like frame. If one fixes conveniently this gauge (using, for example, the Fermi gauge) one obtains a hyperbolic reduction for the evolution equations. As a consequence, given smooth initial data satisfying the constraints, local in time existence and uniqueness of a solution to the EFE can be established —see e.g. [7, 19] and also [24, 25] for details.

A natural way of performing a stability analysis is to consider a sequence of smooth initial data sets \mathbf{u}^ϵ for the EFE satisfying the constraints equations on a Cauchy hypersurface Σ . The sequence is assumed to depend continuously on the parameter ϵ in such a way that the limit $\epsilon \rightarrow 0$ renders the reference solution $\check{\mathbf{u}}$. In particular, one can write the full solution to the EFE as the Ansatz

$$\mathbf{u}^\epsilon = \check{\mathbf{u}} + \epsilon \check{\mathbf{u}}^\epsilon, \quad (2)$$

where $\check{\mathbf{u}}^\epsilon$ is a (nonlinear) perturbation whose size is controlled by the parameter ϵ . Using the Ansatz (2) in equation (1), and writing

$$\begin{aligned} \mathbf{B}(\check{\mathbf{u}} + \epsilon \check{\mathbf{u}}^\epsilon) &= \check{\mathbf{B}}(\check{\mathbf{u}}) + \epsilon \check{\mathbf{B}}(\check{\mathbf{u}}, \epsilon), \\ \mathbf{A}^\mu(\check{\mathbf{u}} + \epsilon \check{\mathbf{u}}^\epsilon) &= \check{\mathbf{A}}^\mu(\check{\mathbf{u}}) + \epsilon \check{\mathbf{A}}^\mu(\check{\mathbf{u}}, \epsilon), \quad \mu = 0, 1, 2, 3 \end{aligned}$$

we are led to consider the following initial value problem for the nonlinear perturbations:

$$\begin{aligned} \left(\check{\mathbf{A}}^0(t, \mathbf{x}, \check{\mathbf{u}}) + \epsilon \check{\mathbf{A}}^0(t, \mathbf{x}, \check{\mathbf{u}}, \epsilon) \right) \partial_t \check{\mathbf{u}} - \left(\check{\mathbf{A}}^j(t, \mathbf{x}, \check{\mathbf{u}}) + \epsilon \check{\mathbf{A}}^j(t, \mathbf{x}, \check{\mathbf{u}}, \epsilon) \right) \partial_j \check{\mathbf{u}} &= \left(\check{\mathbf{B}}(t, \mathbf{x}, \check{\mathbf{u}}) + \epsilon \check{\mathbf{B}}(t, \mathbf{x}, \check{\mathbf{u}}, \epsilon) \right) \check{\mathbf{u}}, \\ \check{\mathbf{u}}(\mathbf{x}, 0) &= \check{\mathbf{u}}_0(\mathbf{x}). \end{aligned} \quad (3)$$

A particular approach to the existence and stability of solutions to the Cauchy problem (3) has been discussed in [8, 9, 21]. In this approach, the stability of solutions follows from the existence of eigenvalues for the nonprincipal part of the linearised system ($\epsilon = 0$) having a negative real part. Stability results for the case where the coefficients of the linearized system are constant matrices are well known [8]. In the case where the system is only strongly hyperbolic, the inner product in L^2 has to be replaced by the so-called \mathcal{H} -inner product —see [9]. A procedure to analyse stability in the case of systems where $\check{\mathbf{B}}$ has vanishing eigenvalues has been given in [9] —see also [10, 21]. These methods can be easily generalized to systems of the type considered here where the matrices $\check{\mathbf{B}}$, $\check{\mathbf{A}}^\mu$ are not constant but depend smoothly on time —see also [20].

The approach described in the previous paragraph has been applied by Reula [20] to the Einstein-perfect fluid system of [6] with a cosmological constant, to prove the exponential decay of nonlinear perturbations for a wide class of homentropic fluids in flat *Friedmann-Robertson-Walker* (FRW) backgrounds. The analysis of [20] gives, as only restriction on the equation of state, that the sound speed is less than one-third of the speed of light. An advantage of this approach is that it avoids the problem of gauge-dependence in perturbation theory.

In the present article we pursue a similar approach to the one used by Reula in [20] to analyse the nonlinear stability of FRW spacetimes with a nonlinear scalar field. To this end, we first construct a first order symmetric hyperbolic system for the EFE with a scalar field as the matter source. This construction is performed by splitting the wave equation for the scalar field into two first order equations. In our analysis, the scalar field is used to construct an adapted orthogonal frame. Written in terms of this adapted frame, the energy-momentum tensor is diagonal, independently of further gauge choices. A similar construction has been considered in the analysis of linear perturbations in [15, 18].

An important difference between the system treated in the present paper and the one in reference [20] is that in our case the matrix $\check{\mathbf{B}}$ has eigenvalues which are zero. These vanishing eigenvalues appear in the block containing the perturbed matter variables and expansion. As it will be discussed in the sequel, this feature is related to the fact that a subspace of the perturbations converge asymptotically to

constant values. Our analysis of the eigenvalues of the nonprincipal part of the linearized system uses results from the so-called *Routh-Hurwitz problem*. In particular, for an ever expanding FRW-nonlinear scalar field background with spatial topology \mathbb{T}^3 , i.e. $([0, \infty) \times \mathbb{T}^3, \dot{\mathbf{g}}, \dot{\phi})$, we show that, for scalar field potentials satisfying certain inequalities, nonlinear perturbations exist and asymptotically exponentially decay to zero or converge to constant values. In the suitable limit, we recover the results of Ringström in [11]. As an application of our results, we show that for power-law solutions produced by an exponential potential with

$$p > \frac{\sqrt{61}-1}{15} \approx 0.454$$

and flat spatial sections, sufficiently small nonlinear perturbations decay exponentially to zero or converge to constant values. We have confirmed this behaviour in some simple numerical computations of linear perturbations. Our analysis shows that nonlinear stability of power-law solutions with exponential potentials is possible even in the absence of accelerated expansion —a possibility that, hitherto, has not been discussed before.

Our main result is proved in the Sobolev norm $H^k(\mathbb{T}^3)$ for $k \geq 5$. As a consequence of the use of results from the Routh-Hurwitz problem, the conditions we obtain on the potential are, in a sense, sharp and cannot be improved by the present methods. Given a particular background Cosmological solution to the Einstein-nonlinear scalar field system —calculated, say, numerically— one can verify explicitly whether the conditions are satisfied or not.

Overview of the article

The paper is organized as follows: in Section 2, we recall Friedrich’s frame formulation of the EFE. In Section 3, we discuss some relevant properties of scalar field satisfying a nonlinear wave equation. In Section 4 we discuss the conditions under which the Einstein-Friedrich-nonlinear scalar field system is well-posed, in the sense that it forms a symmetric hyperbolic system —see Theorem 1. Finally, in Section 5 we give the conditions for which there is convergence to constant values or exponential decay of small nonlinear perturbations on a FRW-nonlinear scalar field background. This is the main result of the paper and we summarise it in Theorem 5. We use units such that $8\pi G = c = 1$.

2 Friedrich’s frame formulation of Einstein Field Equations

In this section we provide a brief introduction to Friedrich’s frame formulation of the Einstein field equations. The basic equation of Friedrich’s construction is the contracted Bianchi identity. From the latter it is possible to deduce hyperbolic propagation equations for the conformal Weyl tensor for a wide spectrum of gauge choices.

2.1 Basic definitions

Let $(\mathcal{M}, \mathbf{g})$ denote a 4-dimensional spacetime with \mathbf{g} a Lorentzian metric of signature $(-, +, +, +)$. In order to implement the frame formulation of the Einstein field equations, one defines locally an orthonormal moving frame or *tetrad* with respect to the metric \mathbf{g} in an open neighbourhood $\mathcal{U} \subset \mathcal{M}$. The frame is a set $\{\mathbf{e}_a\}$ of linearly independent vector fields in the tangent space $T_p(\mathcal{M})$ at each point $p \in \mathcal{U}$ such that

$$\mathbf{g}(\mathbf{e}_a, \mathbf{e}_b) = \eta_{ab}, \quad a, b = 0, \dots, 3, \quad (4)$$

where $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$. The *norm* of a vector field, $\mathbf{v} \in T_p(\mathcal{M})$, in an orthonormal frame is defined as

$$|\mathbf{v}|^2 \equiv \mathbf{g}(\mathbf{v}, \mathbf{v}) = v^a v^b \eta_{ab}$$

and in terms of a coordinate basis set $\{\partial_\alpha\}$ we have $\mathbf{e}_a = e_a^\mu \partial_\mu$. Condition (4) gives

$$\eta_{ab} = e_a^\mu e_b^\nu g_{\mu\nu}.$$

where $\mu, \nu = 0, 1, 2, 3$ denote spacetime tensor indices. The *frame commutator* is written as

$$[\mathbf{e}_a, \mathbf{e}_b] = c_{ab}^c \mathbf{e}_c, \quad (5)$$

where c_{ab}^c are the *structure coefficients*. The *dual basis* or *coframe* is the set of linear forms $\{\theta^b\}$ belonging to the dual space $T_p^*(\mathcal{M})$ at each point $p \in \mathcal{U}$ defined by the pairing $\langle \theta^b, e_a \rangle = \delta_a^b$. In terms of the dual basis we can write condition (4) as

$$g = -(\theta^0)^2 + \sum_{a=1}^3 (\theta^a)^2.$$

The *spacetime (Levi-Civita) connection* in an orthonormal basis is defined by

$$\nabla_a e_b \equiv \gamma_{ba}^c e_c,$$

where γ_{ba}^c are the *connections coefficients*. The covariant derivative of a tensor in \mathcal{M} can be written as

$$\begin{aligned} \nabla_a v_{q_1 \dots q_s}^{p_1 \dots p_r} &= e_a(v_{q_1 \dots q_s}^{p_1 \dots p_r}) + \gamma_{fa}^{p_1} v_{q_1 \dots q_s}^{f \dots p_r} + \dots \\ &\quad \dots + \gamma_{fa}^{p_r} v_{q_1 \dots q_s}^{p_1 \dots f} - \gamma_{q_1 a}^f v_{f \dots q_s}^{p_1 \dots p_r} - \dots - \gamma_{q_s a}^f v_{q_1 \dots f}^{p_1 \dots p_r}. \end{aligned}$$

The torsion free and metric compatibility conditions imply, respectively, that

$$c_{ab}^c = \gamma_{ba}^c - \gamma_{ab}^c, \quad \gamma_{ba}^e \eta_{ec} - \gamma_{ca}^e \eta_{eb} = 0.$$

The equations for the frame coefficients $\{e_a^\mu\}$ are given by equation (5) in terms of the connection coefficients. In turn, equations for the connection coefficients are obtained from the Ricci identity

$$R_{bcd}^a = e_c(\gamma_{bd}^a) - e_d(\gamma_{bc}^a) + \gamma_{fc}^a \gamma_{bd}^f - \gamma_{fd}^a \gamma_{bc}^f - \gamma_{bf}^a (\gamma_{dc}^f - \gamma_{cd}^f). \quad (6)$$

The Riemann tensor can be decomposed in terms of the conformal Weyl tensor \mathbf{C} and the Schouten tensor \mathbf{S} as

$$R_{bcd}^a = C_{bcd}^a + \delta_{[c}^a S_{d]b} - \eta_{b[c} S_{d]}^a. \quad (7)$$

For future use we introduce the *Friedrich tensor* \mathbf{F} via

$$F_{abcd} \equiv C_{abcd} - \eta_{a[c} S_{d]b}, \quad (8)$$

and its dual with respect to the last pair of indices

$${}^*F_{abcd} = {}^*C_{abcd} + \frac{1}{2} S_{pb} \epsilon^p{}_{acd}, \quad (9)$$

where ϵ_{abcd} is the usual Levi-Civita totally antisymmetric symbol with $\epsilon_{0123} = 1$. In terms of the Friedrich tensor, one finds that the contracted Bianchi identities read

$$\nabla_a F_{bcd}^a = 0, \quad \nabla_a {}^*F_{bcd}^a = 0. \quad (10)$$

2.2 Orthonormal decomposition of the field equations

The field equations of Friedrich's frame formulation of the Einstein equations are given by equations (5), (6) and (10), together with the decomposition (7). The independent variables of the system are

$$(e_a^\mu, \gamma_{bc}^a, C_{bcd}^a, S_{bc}).$$

In what follows, we will shall decompose the field equations and relevant tensors in terms of their parallel and orthogonal components with respect to the time-like frame. We write $\mathbf{N} \equiv e_0$ and set

$$\mathbf{N} = N^a e_a, \quad N^a = \delta_0^a,$$

where $N_a = -\delta_a^0$ in our signature. In terms of these objects, tensor fields which are orthogonal to the timelike frame-vector are defined by

$$T_{a_1 \dots a_p \dots a_q} N^{a_p} = 0, \quad p = 1, 2, \dots, q.$$

Next, one defines

$$h_{ab} \equiv \eta_{ab} + N_a N_b,$$

where the projector onto the orthogonal 3-subspaces satisfies $h_a{}^c = \eta^{bc} h_{ab}$. The *spatial covariant derivative* is then given by

$$D_a T_{q_1 \dots q_r} = h_a{}^b h_{q_1}{}^{p_1} \dots h_{q_r}{}^{p_r} \nabla_b T_{p_1 \dots p_r}.$$

In particular, one has that

$$D_a h_{bd} = 0, \quad D_a \epsilon_{bcd} = 0,$$

where ϵ_{bcd} is the spatial Levi-Civita symbol and the indices run from 1 to 3.

In order to further proceed with the geometric decomposition one defines the *acceleration vector* by

$$\mathbf{a} \equiv \nabla_0 \mathbf{e}_0 = \gamma^p{}_{00} \mathbf{e}_p, \quad p = 1, 2, 3.$$

It follows then that $a^p = \gamma^p{}_{00}$ or equivalently, $a_p = \gamma^0{}_{p0}$. We will also consider the so-called *Weingarten map* given by

$$\chi(\mathbf{e}_a) \equiv \nabla_a \mathbf{e}_0 = \gamma^p{}_{0a} \mathbf{e}_p, \quad a, p = 1, 2, 3.$$

It follows then that $\chi_a{}^p = \gamma^p{}_{0a}$. The tensor χ_{ab} can be written in terms of its irreducible parts as

$$\chi_{ab} = \gamma^0{}_{ba} = (\chi^{ST})_{ab} + \frac{1}{3} \chi h_{ab} + (\chi^A)_{ab},$$

where $(\chi^{ST})_{ab}$, χ , $(\chi^A)_{ab}$ denote, respectively, its symmetric trace-free, trace and antisymmetric parts. If the flow of \mathbf{e}_0 is hypersurface orthogonal then one has that $(\chi^A)_{ab} = 0$ and that

$$\frac{1}{2} \mathcal{L}_N h_{ab} = \chi_{(ab)} = (\chi^{ST})_{ab} + \frac{1}{3} \chi h_{ab}, \quad (11)$$

where \mathcal{L}_N denotes the Lie derivative along \mathbf{N} and $\nabla_a N^p = -N_a a^p + \chi_a{}^p$.

Finally, the 4-dimensional Levi-Civita symbol is also decomposed using

$$\epsilon_{abcd} = 2\epsilon_{ab[c} N_{d]} - 2N_{[a} \epsilon_{b]cd}.$$

Now, defining $\tilde{F}_{bcd} \equiv \nabla_a F^a{}_{bcd}$ it follows that the first contracted Bianchi identity can be written as

$$\tilde{F}_{bcd} = N_b \left[\tilde{F}'_{0c0} N_d - \tilde{F}'_{0d0} N_c \right] + 2\tilde{F}'_{b0[c} N_{d]} - N_b \tilde{F}'_{0cd} + \tilde{F}'_{bcd} = 0, \quad (12)$$

where contractions with \mathbf{N} are denoted by the index 0, and the prime ' indicates that the remaining indices are spatial. For example, $\tilde{F}'_{b0d} \equiv h_b{}^q N^r h_d{}^s \tilde{F}_{qrs}$.

Given the vector \mathbf{N} , the Weyl tensor is uniquely determined through its *electric* and *magnetic* parts. These are defined, respectively, by

$$E_{ab} \equiv h_a{}^p h_b{}^q N^c N^d C_{pqcd}, \quad B_{bd} \equiv h_b{}^p h_d{}^q N^a N^c C_{apcq}^*.$$

In terms of the latter, the Weyl tensor and its dual can be written as

$$C_{abcd} = 2[l_{a[c} E_{d]b} - l_{b[c} E_{d]a}] - 2[N_{[c} B_{d]p} \epsilon^p{}_{ab} + N_{[a} B_{b]p} \epsilon^p{}_{cd}] \quad (13)$$

$${}^* C_{abcd} = 2N_{[a} E_{b]p} \epsilon^p{}_{cd} - 4E_{p[a} \epsilon_{b]}{}^p{}_{[c} N_{d]} - 4N_{[a} B_{b][c} N_{d]} - B_{pq} \epsilon^p{}_{ab} \epsilon^q{}_{cd} \quad (14)$$

where $l_{ab} \equiv h_{ab} + N_a N_b$.

3 Nonlinear Scalar fields in the frame formalism

In this section, we introduce a description on nonlinear scalar fields which is particularly well adapted to the analysis to be carried out in the present article.

3.1 Basic equations

In what follows we will consider scalar fields $\phi \in C^\infty(\mathcal{M})$ with a smooth positive selfinteracting potential $\mathcal{V}(\phi)$, and energy-momentum tensors of the form

$$\mathbf{T} = \boldsymbol{\psi} \otimes \boldsymbol{\psi} - \left(\frac{1}{2} |\boldsymbol{\psi}|^2 + \mathcal{V}(\phi) \right) \mathbf{g},$$

where we have defined the 1-form

$$\boldsymbol{\psi} \equiv \nabla \phi$$

with ∇ the spacetime connection. Accordingly, we define

$$\psi_a \equiv \boldsymbol{\psi}(\mathbf{e}_a) = (\psi, \psi'_a), \quad (15)$$

where we have written

$$\psi \equiv \psi_0 = \mathcal{L}_N \phi \quad (16)$$

and

$$\psi'_a \equiv h_a{}^b \psi_b = D_a \phi. \quad (17)$$

The components of the energy-momentum tensor \mathbf{T} with respect to the tetrad $\{\mathbf{e}_a\}$ are given by

$$T_{ab} = \psi_a \psi_b - \left(\frac{1}{2} |\boldsymbol{\psi}|^2 + \mathcal{V}(\phi) \right) \eta_{ab}, \quad (18)$$

while its trace is

$$T = -|\boldsymbol{\psi}|^2 - 4\mathcal{V}(\phi).$$

The Einstein field equations in the form

$$\mathbf{R} = \mathbf{T} - \frac{1}{2} \mathbf{g} (\text{Tr} \mathbf{T}),$$

then imply for the components of the Ricci tensor that

$$R_{ab} = \psi_a \psi_b + \mathcal{V}(\phi) \eta_{ab},$$

while the Ricci scalar is given by

$$R = -T = |\boldsymbol{\psi}|^2 + 4\mathcal{V}(\phi).$$

From these expressions it follows that the components of the Schouten tensor with respect to the frame $\{\mathbf{e}_a\}$ are given by

$$S_{ab} = \psi_a \psi_b - \frac{1}{3} \left(\frac{1}{2} |\boldsymbol{\psi}|^2 - \mathcal{V}(\phi) \right) \eta_{ab}.$$

3.2 Gauge considerations

In order to construct and adapted frame to our particular problem, we let $\boldsymbol{\psi} \equiv \alpha \mathbf{e}_0$. It follows that

$$\psi^a = \alpha \delta_0^a, \quad (19)$$

so that

$$\alpha = -\psi \quad \text{and} \quad D^a \phi = 0.$$

Accordingly,

$$|\boldsymbol{\psi}|^2 = \mathbf{g}(\boldsymbol{\psi}, \boldsymbol{\psi}) = \alpha^2 \eta_{00} = -\alpha^2, \quad \alpha = \pm \sqrt{-|\boldsymbol{\psi}|^2}. \quad (20)$$

If the vector $\boldsymbol{\psi}$ is taken to be future oriented, then one must choose α to be positive. In terms of a coordinate basis the latter implies

$$\psi^\mu = \alpha e_0^\mu = -\psi e_0^\mu, \quad e_0^\mu = \frac{\nabla^\mu \phi}{\sqrt{-|\boldsymbol{\psi}|^2}}. \quad (21)$$

and

$$D_a \phi = 0, \quad e_a{}^\mu \nabla_\mu \phi = 0. \quad (22)$$

With this choice, we have that

$$\psi_a = -\psi N_a,$$

and therefore

$$T_{ab} = \left(\frac{1}{2} \psi^2 + \mathcal{V}(\phi) \right) N_a N_b + \left(\frac{1}{2} \psi^2 - \mathcal{V}(\phi) \right) h_{ab}, \quad (23)$$

$$S_{ab} = \frac{1}{3} \left(\frac{5}{2} \psi^2 - \mathcal{V}(\phi) \right) N_a N_b + \frac{1}{3} \left(\frac{1}{2} \psi^2 + \mathcal{V}(\phi) \right) h_{ab}. \quad (24)$$

Using (15) and (20), the equation for the conservation of the energy-momentum tensor takes the form:

$$\nabla^a T_{ab} = \nabla^a \left(\psi^2 N_a N_b + \left(\frac{1}{2} \psi^2 - \mathcal{V}(\phi) \right) \eta_{ab} \right) \quad (25)$$

$$= 2\psi N_b N^a (\nabla_a \psi) + \psi^2 (N_b (\nabla_a N^a) + N^a (\nabla_a N_b)) + \nabla_b \left(\frac{1}{2} \psi^2 - \mathcal{V}(\phi) \right) \quad (26)$$

$$= \left(2\psi \mathcal{L}_N \psi + \psi^2 \chi + \psi \frac{d\mathcal{V}}{d\phi} \right) N_b + \psi^2 a_b + \psi \nabla_b \psi = 0. \quad (27)$$

From the latter, projecting with respect to the timelike frame one obtains:

$$N^b (\nabla^a T_{ab}) = 0, \quad \mathcal{L}_N \psi + \chi \psi + \frac{d\mathcal{V}}{d\phi} = 0, \quad (28)$$

$$h_c{}^b (\nabla^a T_{ab}) = 0, \quad D_c \psi + \psi a_c = 0. \quad (29)$$

Moreover, using that $D_a \phi = 0$ in the orthogonal subspaces to \mathbf{e}_0 , one obtains from equation (5) that

$$[\mathbf{e}'_a, \mathbf{e}'_b] \phi = 2 (\chi^A)_{ab} \psi = 0,$$

which implies that

$$(\chi^A)_{ab} = 0. \quad (30)$$

Remark. Following Friedrich in [7], one could as well have defined

$$\nabla^a T_{ab} = q_b + q N_b, \quad J_{ab} = \nabla_{[a} q_{b]}. \quad (31)$$

Then, instead of using the condition on the vanishing of the divergence of the energy-momentum tensor, one could include the equations $q = 0$ and $q_b = 0$ as a part of the equations determining the Einstein-scalar field system in the frame representation. Once the gauge is fixed, the first equation in (31) appears in the reduced system of evolution equations while the second part is regarded as a *zero quantity*—see equation (4.44) in [7]. It can be shown that the zero quantities satisfy a system of *subsidiary evolution equations*. For this, it can be shown that the zero quantities vanish if they are zero on the initial hypersurface. For the quantity q_b the relevant subsidiary equation is given in equation (4.70) of [7]. We also notice that the evolution for the acceleration can be computed from the tensor J_{ab} .

4 The Einstein-Friedrich-nonlinear scalar field system

In this section, we derive a first order symmetric hyperbolic system for the EFE coupled to a nonlinear scalar field. Making use of the Bianchi identity and the energy-momentum tensor of equation (18), we derive the propagation equations for the *electric* and *magnetic* parts of the conformal Weyl tensor. After fixing the gauge, we complete the reduced system of evolution equations by deriving equations for the frame and the connection coefficients. In the last part of this section we make some remarks concerning the hyperbolicity of the system.

4.1 Basic expressions

We start by computing the various components for the Friedrich tensor \mathbf{F} . Using equations (13) and (24), one finds that the definition (8) implies

$$\begin{aligned}
F'_{00c0} &= 0 = -F'_{000c}, & F'_{00cd} &= 0 = -F'_{00dc}, \\
F'_{a00d} &= -E_{ad} + \frac{1}{6} \left(\frac{5}{2} \psi^2 - \mathcal{V}(\phi) \right) h_{ad} = -F'_{a0d0}, \\
F'_{ab0d} &= B_{dp} \epsilon^p_{ab} = -F'_{abd0} = -F'_{ba0d}, \\
F'_{0bcd} &= B_{bp} \epsilon^p_{cd} = -F'_{0bdc} = -F'_{b0cd}, \\
F'_{0b0d} &= E_{bd} + \frac{1}{6} \left(\frac{1}{2} \psi^2 + \mathcal{V}(\phi) \right) h_{bd} = -F'_{0bd0}, \\
F'_{abcd} &= -2 (h_{b[c} E_{d]a} - h_{a[c} E_{d]b}) - \frac{1}{6} \left(\frac{1}{2} \psi^2 + \mathcal{V}(\phi) \right) (h_{ac} h_{db} - h_{ad} h_{cb}),
\end{aligned} \tag{32}$$

with the nonvanishing traces

$$\begin{aligned}
h^{ac} F'_{a0c0} &= \frac{1}{2} \left(\frac{5}{2} \psi^2 - \mathcal{V}(\phi) \right), \\
h^{bd} F'_{0b0d} &= \frac{1}{2} \left(\frac{1}{2} \psi^2 + \mathcal{V}(\phi) \right) = -h^{bd} F'_{0bd0}, \\
h^{bd} F'_{abcd} &= E_{ac} - \frac{1}{3} \left(\frac{1}{2} \psi^2 + \mathcal{V}(\phi) \right) h_{ac} = F'^b{}_{abc}, \\
h^{ac} h^{bd} F'_{abcd} &= -\frac{1}{2} \psi^2 - \mathcal{V}(\phi).
\end{aligned} \tag{33}$$

Using the expression (9) with equations (14) and (24), we get the following components of the dual ${}^* \mathbf{F}$:

$$\begin{aligned}
{}^* F'_{00c0} &= -{}^* F'_{000c} = 0, & {}^* F'_{00cd} &= -{}^* F'_{00dc} = 0, \\
{}^* F'_{a0c0} &= -{}^* F'_{a00c} = B_{ac}, \\
{}^* F'_{abc0} &= -2 E_{p[b} \epsilon_{a]}^p{}_c - \frac{1}{6} \left(\frac{1}{2} \psi^2 + \mathcal{V}(\phi) \right) \epsilon_{bac} = -{}^* F'_{ab0c}, \\
{}^* F'_{a0cd} &= E_{ap} \epsilon^p_{cd} - \frac{1}{6} \left(\frac{5}{2} \psi^2 - \mathcal{V}(\phi) \right) \epsilon_{acd}, \\
{}^* F'_{0bcd} &= -E_{bp} \epsilon^p_{cd} - \frac{1}{6} \left(\frac{1}{2} \psi^2 + \mathcal{V}(\phi) \right) \epsilon_{bcd}, \\
{}^* F'_{0b0d} &= B_{bd}, & {}^* F'_{abcd} &= -B_{pq} \epsilon^p_{ab} \epsilon^q_{cd}.
\end{aligned} \tag{34}$$

4.2 The Bianchi equations

If one substitutes the expressions for the Friedrich tensor derived in the previous section into the first Bianchi identities (12) one obtains the following relations for the components of the zero quantity \tilde{F}_{abc} :

$$\begin{aligned}
\tilde{F}'_{0c0} &= -\mathcal{L}_N F'_{00c0} + D^q F'_{q0c0} + \chi_c {}^s F'_{00s0} - \chi F'_{00c0} - \chi^{qb} (F'_{qb0c} + F'_{q0cb}) + a^b (F'_{0bc0} + F'_{00cb} + F'_{b0c0}), \\
\tilde{F}'_{0cd} &= -\mathcal{L}_N F'_{00cd} + D^q F'_{q0cd} + a^b F'_{0bcd} + a^q F'_{q0cd} - \chi^{qb} F'_{qbcd} - \chi F'_{00cd} - \chi^q{}_c F'_{q00d} - \chi^q{}_d F'_{q0c0} \\
&\quad + \chi_c {}^s F'_{00sd} + \chi_d {}^s F'_{00cs} + a_c F'_{000d} + a_d F'_{00c0}, \\
\tilde{F}'_{b0d} &= -\mathcal{L}_N F'_{0b0d} + D^a F'_{ab0d} - \chi F'_{0b0d} - \chi^a{}_b F'_{a00d} - \chi^{ac} F'_{abcd} + \chi_b {}^s F'_{0s0d} + a_b F'_{000d} + \chi_d {}^s F'_{0b0s} \\
&\quad + a^q F'_{qb0d} + a^c F'_{0bcd}, \\
F'_{bcd} &= -\mathcal{L}_N F'_{0bcd} + D^a F'_{ab0d} + a^q F'_{qbcd} + (a_b F'_{00cd} + a_c F'_{0b0d} + a_d F'_{0bc0}) - \chi F'_{0bcd} \\
&\quad + \chi_b {}^q F'_{0qcd} + \chi_c {}^q F'_{0bqd} + \chi_d {}^q F'_{0bcq} - \chi^q{}_b F'_{q0cd} - \chi^q{}_c F'_{qb0d} - \chi^q{}_d F'_{qbc0},
\end{aligned} \tag{35}$$

where we have used the fact that \mathbf{F} is anti-symmetric in the last two indices —see e.g. [6]. Similar relations hold for the dual ${}^* \tilde{\mathbf{F}}$.

Remark. In reference [7] —cfr. equation (4.47)— suitable zero quantities are defined by using the decomposition in terms of irreducible components of $\tilde{\mathbf{F}}$.

4.2.1 The evolution equation for the electric part of the Weyl tensor

An evolution equation for the electric part of the Weyl tensor can be obtained using the third equation of (35) together with the expressions (32) and (33), and then symmetrising with respect to the indices (bd) . One obtains the equation

$$\begin{aligned}\tilde{F}'_{(b|0|d)} = & -\mathcal{L}_N E_{bd} - \frac{1}{6} \left(\frac{1}{2} \psi^2 + \mathcal{V}(\phi) \right) \mathcal{L}_N h_{bd} - \frac{1}{6} h_{bd} \mathcal{L}_N \left(\frac{1}{2} \psi^2 + \mathcal{V}(\phi) \right) + D_a B_{p(d\epsilon_b)}^{pa} \\ & + 2a_a B_{p(b\epsilon_d)}^{pa} - 2\chi E_{bd} + 2\chi^a_{(b} E_{d)q} + 3\chi_{(b}{}^q E_{d)q} - h_{db} \chi^{ac} E_{ac} - \frac{1}{3} (\psi^2 - \mathcal{V}(\phi)) \chi_{(bd)}.\end{aligned}$$

Similarly, using equation (11) we get

$$\begin{aligned}\tilde{F}'_{(b|0|d)} = & -\mathcal{L}_N E_{bd} + D_a B_{p(d\epsilon_b)}^{pa} + 2a_a B_{p(b\epsilon_d)}^{pa} - 2\chi E_{bd} + 2\chi^a_{(b} E_{d)q} + 3\chi_{(b}{}^q E_{d)q} - h_{db} \chi^{ac} E_{ac} \\ & - \frac{1}{2} \psi^2 \chi_{(bd)} - \frac{1}{6} h_{bd} \mathcal{L}_N \left(\frac{1}{2} \psi^2 + \mathcal{V}(\phi) \right).\end{aligned}$$

The trace of the previous expression is given by

$$h^{rs} \tilde{F}'_{(r|0|s)} = -\frac{1}{2} \psi^2 \chi - \frac{1}{2} \mathcal{L}_N \left(\frac{1}{2} \psi^2 + \mathcal{V}(\phi) \right),$$

which is the evolution equation for the scalar field —i.e. the equation expressing the conservation of energy. The evolution equation for the components of the tensor E_{ab} is obtained by taking the difference of the last two equations so that:

$$\begin{aligned}\mathcal{L}_N E_{bd} - D_a B_{p(d\epsilon_b)}^{pa} = & 2a_a B_{p(b\epsilon_d)}^{pa} - 2\chi E_{bd} + 2\chi^q_{(b} E_{d)q} + 3\chi_{(b}{}^q E_{d)q} - h_{db} \chi^{ac} E_{ac} \\ & - \frac{1}{2} \psi^2 \left(\chi_{(bd)} - \frac{1}{3} \chi h_{bd} \right).\end{aligned}$$

Finally using (30) we can write the last equation as

$$\mathcal{L}_N E_{bd} - D_a B_{p(d\epsilon_b)}^{pa} = -\frac{1}{2} \psi^2 (\chi^{ST})_{bd} - \frac{1}{3} \chi E_{bd} + 5 (\chi^{ST})^q_{(b} E_{d)q} + 2a_a B_{p(b\epsilon_d)}^{pa}. \quad (36)$$

4.2.2 The evolution equation for the magnetic part of the Weyl tensor

An evolution equation for the magnetic part of the Weyl tensor can be derived from the third equation of (35) using the expressions (34). A computation yields

$$\begin{aligned}\tilde{F}_{b0d} = & -\mathcal{L}_N B_{bd} + D^a \left(2E_{p[b\epsilon_a]}^p{}_d + \frac{1}{6} \left(\frac{1}{2} \psi^2 + \mathcal{V}(\phi) \right) \epsilon_{bad} \right) - \chi B_{bd} + \chi^a{}_b B_{ad} + 2\chi_{(b}{}^a B_{d)a} \\ & + 2a_a B_{pb} \epsilon^{qp}{}_d + a^q E_{pq} \epsilon^p{}_{bd} + \chi^{qb} B_{pq} \epsilon^p{}_{ab} \epsilon^q{}_{cd}.\end{aligned}$$

Now, since B_{bd} is a symmetric tensor, all the information about its evolution is contained in the symmetrised expression $\tilde{F}_{(b|0|d)}$. Consequently, symmetrising the previous equation with respect to the spatial indices (bd) we find

$$\mathcal{L}_N B_{bd} - D_a E_{p(b\epsilon_d)}^{ap} = -2a_a E_{p(b\epsilon_d)}^{pa} + 2\chi_{(b}{}^a B_{d)a} + \chi^a{}_{(d} B_{b)a} - \chi B_{bd} + \chi_{ac} B_{pq} \epsilon^{pa}{}_{(b\epsilon_d)}{}^{qc},$$

so that using (30) we have

$$\partial_t B_{bd} - D_a E_{p(b\epsilon_d)}^{ap} = -\frac{1}{3} \chi B_{bd} + 3 (\chi^{ST})^q_{(b} B_{d)q} + (\chi^{ST})_{ac} B_{pq} \epsilon^{pa}{}_{(b\epsilon_d)}{}^{qc} - 2a_a E_{p(b\epsilon_d)}^{pa}. \quad (37)$$

The principal part of the evolution equations (36) and (37) is given by the symmetric matrix

$$\begin{pmatrix} \mathbf{e}_0 & 0 & 0 & 0 & -\frac{1}{2} D_1 & \frac{1}{2} D_2 \\ 0 & \mathbf{e}_0 & 0 & \frac{1}{2} D_1 & 0 & -\frac{1}{2} D_3 \\ 0 & 0 & \mathbf{e}_0 & -\frac{1}{2} D_2 & \frac{1}{2} D_3 & 0 \\ 0 & \frac{1}{2} D_1 & -\frac{1}{2} D_2 & \mathbf{e}_0 & 0 & 0 \\ -\frac{1}{2} D_1 & 0 & \frac{1}{2} D_3 & 0 & \mathbf{e}_0 & 0 \\ \frac{1}{2} D_2 & -\frac{1}{2} D_3 & 0 & 0 & 0 & \mathbf{e}_0 \end{pmatrix}.$$

4.3 The Lagrangian description and Fermi transport

In order to deduce the remaining evolution equations, we will adopt a Lagrangian description. This point of view amounts to requiring the timelike vector of the orthonormal frame to follow the matter flow lines. Accordingly, we introduce coordinates (t, \mathbf{x}) such that

$$\mathbf{e}_0 = \partial_t, \quad e_0^\mu = \delta_0^\mu. \quad (38)$$

This particular choice is equivalent to setting $\theta^b = \theta^b_j dx^j$ while at the same time fixing the lapse function to one¹. With this choice (since $\mathcal{L}_N = \partial_t$), we have from equations (16), (28), and (22) that

$$\partial_t \phi = \psi \equiv N^a \psi_a = -\alpha < 0, \quad (39)$$

$$\partial_t \psi = -\psi \chi - \frac{d\mathcal{V}}{d\phi}, \quad (40)$$

$$e_a'^0 \psi = -e_a'^j D_j \phi. \quad (41)$$

Now, the timelike coframe is given in terms of the natural cobasis through the relation

$$\theta^0 = dt + \beta_j dx^j,$$

while the spatial frame vectors are found to be

$$\mathbf{e}'_a = (\theta_a^j)^{-1} (\partial_j - \beta_j \partial_t), \quad e_a'^0 = (\theta_a^j)^{-1} \beta_j, \quad e_a'^j = (\theta_a^j)^{-1}. \quad (42)$$

It then follows from equations (41) and (42) that

$$\beta_j = -\frac{1}{\psi} D_j \phi. \quad (43)$$

Thus, since β_j is nonzero, the surfaces of constant time are not necessarily spacelike for the characteristic cone and this could be a problem for the hyperbolicity of the system —see [24, 25].

Finally, the remaining frame components are chosen to be Fermi propagated along \mathbf{e}_0 . That is, we require

$$\nabla_0 \mathbf{e}_a - (\mathbf{g}(\mathbf{e}_a, \nabla_0 \mathbf{e}_0) \mathbf{e}_0 - \mathbf{g}(\mathbf{e}_a, \mathbf{e}_0) \nabla_0 \mathbf{e}_0) = 0.$$

The latter condition implies

$$\gamma'^a_{b0} = 0.$$

4.4 Evolution equation for the frame coefficients

As already mentioned, the evolution equations are obtained from the relation (5) to yield

$$[e_0, \mathbf{e}'_b] = a_b \mathbf{e}_0 - \gamma'^c_{0b} \mathbf{e}'_c,$$

where it has been used that $\gamma'^c_{b0} = 0$ (Fermi gauge). Therefore, the evolution equations for the remaining frame coefficients read

$$\begin{aligned} \partial_t e_b'^i &= -\chi_b^c e_c'^i, \\ \partial_t e_b'^0 &= a_b - \chi_b^c e_c'^0, \end{aligned}$$

which, together with (42), imply propagation equations for the components of the metric in the local coordinate system. In particular, one has that

$$\partial_t \beta_j = \theta'^b_j a_b$$

with β_j given by equation (43) —see also equation (6.2) in [24] for arbitrary lapse U . Finally, since from (30) we have $(\chi^A)_{ab} = 0$, the evolution equations for the frame coefficients can be written as

$$\begin{aligned} \partial_t e_b'^i &= -\frac{1}{3} \chi e_b'^i - (\chi^{ST})_b^c e_c'^i \\ \partial_t e_b'^0 &= -\frac{1}{3} \chi e_b'^0 - (\chi^{ST})_b^c e_c'^0 + a_b. \end{aligned} \quad (44)$$

¹See also [24], where a symmetric hyperbolic system was obtained for the Einstein-Euler system. This construction holds for an arbitrary Eulerian picture.

4.5 Evolution equations for the connection coefficients

The equations for the connection coefficients are obtained from the splitting of the Riemann tensor with respect to the frame $\{e_a\}$. In general, we have that

$$\begin{aligned} R'^a{}_{b0d} &= e_0(\gamma^a{}_{bd}) - D_d\gamma^a{}_{b0} - a_d\gamma^a{}_{b0} - (\gamma^p{}_{d0} - \chi_d{}^p)\gamma'^a{}_{bp} - a_b\chi_d{}^a + a^a\chi_{db}, \\ R'^0{}_{b0d} &= e_0(\chi_{db}) - D_da_b - a_ba_d + \chi_{pb}\chi_d{}^p - \chi_{dp}\gamma^p{}_{b0} - \chi_{pb}\gamma^p{}_{d0}, \\ R'^a{}_{0cd} &= D_c\chi_d{}^a - D_d\chi_c{}^a - a^a(\chi_{cd} - \chi_{dc}), \\ R'^a{}_{bcd} &= \tilde{R}^a{}_{bcd} + \chi_c{}^a\chi_{db} - \chi_d{}^a\chi_{cb} - \gamma^a{}_{b0}(\chi_{cd} - \chi_{dc}), \end{aligned} \quad (45)$$

where $\tilde{R}^a{}_{bcd}$ denotes the Riemann tensor constructed only with the spatial connection coefficients $\gamma'^c{}_{ab}$. The first two identities give evolution equations once the Lagrangian gauge is introduced. The remaining two equations are the *quasi-constraints* for the connection coefficients —see [24, 25]. No equations for the connection coefficient associated to the acceleration can be deduced from these identities. In the sequel, it will be shown how evolution equations for the acceleration can be obtained for our particular problem.

From equations (45), we can also deduce two important equations relating the Ricci tensor to the connection:

$$\begin{aligned} R_{00} &= -e_0(\chi) + D_p a^p - \chi_p{}^b \chi_b{}^p + a_p a^p \\ R'_{0d} &= D_c \chi_d{}^c - D_d \chi - 2a^c(\chi^A)_{cd}. \end{aligned} \quad (46)$$

The first identity in (45), together with the conditions for the Lagrangian and Fermi gauge provide the equation

$$\partial_t \gamma^a{}_{bd} = -\gamma^a{}_{bp} \chi_d{}^p + 2h^{ap} \chi_{d[p} a_{b]} + B_{dp} \epsilon^{pa}{}_{b},$$

describing the evolution of the spatial connection coefficients $\gamma'^c{}_{ab}$. To obtain the last equation it has been used that $R'^a{}_{b0d} = C'^a{}_{b0d} = B_{dp} \epsilon^{pa}{}_{b}$. Moreover, since from (30) we have $(\chi^A)_{ab} = 0$, we can write

$$\partial_t \gamma^a{}_{bd} = -\frac{1}{3} \chi \gamma^a{}_{bd} + \frac{2}{3} \chi h^{ap} h_{d[p} a_{b]} + B_{dp} \epsilon^{pa}{}_{b} - (\chi^{ST})_d{}^p \gamma^a{}_{bp} + 2h^{ap} (\chi^{ST})_{d[p} a_{b]}. \quad (47)$$

The evolution equation for the part of the connection described by χ_{bd} is obtained from the second identity in (45). In order to do so, first, we will derive the evolution and the quasi-constraint equations for the acceleration. The evolution equation for the acceleration can be obtained from

$$\begin{aligned} [e_0, e'_c] \psi &= c^0{}_{0c} e_0(\psi) + c^p{}_{0c} e'_p(\psi) \\ &= \gamma^0{}_{c0} (\partial_t \psi) + (\gamma^p{}_{c0} - \gamma^p{}_{0c}) (D_p \psi) \\ &= a_c (\partial_t \psi) - \chi_c{}^p (D_p \psi), \end{aligned}$$

where the Lagrangian and Fermi gauge have been employed. Now, expanding the left hand side and making use of the evolution and the quasi-constraint equation for the energy-momentum tensor of the scalar field one has that

$$\partial_t a_c - D_c \chi = -\chi_c{}^p a_p + \left(\chi + \frac{2}{\psi} \frac{d\mathcal{V}}{d\phi} \right) a_c$$

so that using the second equation in (46) we arrive at

$$\partial_t a_c - D_p \chi_c{}^p = -2(\chi^A)_{pc} a^p - \chi_c{}^p a_p + \left(\chi + \frac{2}{\psi} \frac{d\mathcal{V}}{d\phi} \right) a_c. \quad (48)$$

Finally, since for our particular problem one has that $(\chi^A)_{ab} = 0$, this last equation takes the form

$$\partial_t a_c - D_p \chi_c{}^p = \left(\frac{2}{\psi} \frac{d\mathcal{V}}{d\phi} + \frac{2}{3} \chi \right) a_c - (\chi^{ST})_c{}^b a_b. \quad (49)$$

In the case of the quasi-constraint a computation yields

$$D_c a_b - D_b a_c = 2 \left(\chi + \frac{1}{\psi} \frac{d\mathcal{V}}{d\phi} \right) (\chi^A)_{cb}. \quad (50)$$

Thus, making use of this equation in the second identity in equation (45) and recalling the properties of the Fermi gauge one finds that

$$\partial_t \chi_{db} - D_b a_d = E_{db} + \frac{1}{3} (\mathcal{V}(\phi) - \psi^2) h_{db} - \chi_d^p \chi_{pb} + 2 \left(\chi + \frac{1}{\psi} \frac{d\mathcal{V}}{d\phi} \right) (\chi^A)_{db} + a_d a_b, \quad (51)$$

where it has been used

$$R'^0{}_{b0d} = C'^0{}_{b0d} + \frac{1}{2} \delta_0^0 S_{bd} + \frac{1}{2} h_{bd} S_0^0 = E_{bd} + \frac{1}{3} (\mathcal{V}(\phi) - \psi^2) h_{bd}.$$

The principal part of the combined system of equations (48) and (51) is given by

$$\begin{pmatrix} \mathbf{e}_0 & -D_1 & -D_2 & -D_3 \\ -D_1 & \mathbf{e}_0 & 0 & 0 \\ -D_2 & 0 & \mathbf{e}_0 & 0 \\ -D_3 & 0 & 0 & \mathbf{e}_0 \end{pmatrix} \begin{pmatrix} a_d \\ \chi_d^1 \\ \chi_d^2 \\ \chi_d^3 \end{pmatrix}.$$

Since $(\chi^A)_{ab} = 0$, then χ_{bd} is symmetric and in this case, we have after symmetrisation that

$$2\partial_t \chi_{(bd)} - 2D_{(b} a_{d)} = \frac{2}{3} \left(\mathcal{V}(\phi) - \psi^2 - \frac{1}{3} \chi^2 \right) h_{bd} - \frac{4}{3} \chi (\chi^{ST})_{bd} - 2 (\chi^{ST})_b{}^p (\chi^{ST})_{pd} + 2a_b a_d + 2E_{bd}. \quad (52)$$

Also, from the first equation in (46) it follows that

$$\partial_t \chi - D_p a^p = \mathcal{V}(\phi) - \psi^2 - \frac{\chi^2}{3} - (\chi^{ST})^2 + a^2,$$

where $(\chi^{ST})^2 = (\chi^{ST})_{ab} (\chi^{ST})^{ab}$ and $a^2 = a_c a^c$.

4.6 Hyperbolicity considerations

The system consisting of equations (39), (40), (36), (37), (44), (47), (49) and (52) can be written matrixially as

$$\mathbf{A}^0 \partial_t \mathbf{u} - \mathbf{A}^p \mathbf{e}'_p(\mathbf{u}) = \mathbf{B}(\mathbf{u}) \mathbf{u}. \quad (53)$$

As discussed in [24], these systems are not hyperbolic in the usual sense as, in general, the time lines are not hypersurface orthogonal and the “spatial” frame vectors \mathbf{e}'_a have components in the time direction —see (42). Since the surfaces of constant time t are not necessarily spacelike, this type of system is referred to as a *quasi FOSH* system —see [24]. In terms of the partial derivatives, equation (53) reads

$$\tilde{\mathbf{A}}^0(\mathbf{u}) \partial_t \mathbf{u} - \mathbf{A}^j(\mathbf{u}) \partial_j \mathbf{u} = \mathbf{B}(\mathbf{u}) \mathbf{u} \quad (54)$$

with

$$\tilde{\mathbf{A}}^0(\mathbf{u}) \equiv \mathbf{A}^0 - \mathbf{A}^p \mathbf{e}'_p{}^0, \quad \mathbf{A}^j(\mathbf{u}) \equiv \mathbf{A}^p \mathbf{e}'_p{}^j. \quad (55)$$

In order to have a well posed initial value problem, the matrix $\tilde{\mathbf{A}}^0(\mathbf{u})$ must be positive definite. This is the case as long the quadratic form

$$\sum_{b=1,2,3} \theta^b{}_i \theta^b{}_j - \beta_i \beta_j \quad (56)$$

is positive definite —see *Proposition 9* in [24]. In the next section we will consider a reference solution admitting a foliation by homogeneous spacelike hypersurfaces. As a consequence, the linearisation of the system (3) is well posed without the need to control the smallness of β_i . The smallness of these terms is taken care by the perturbation fields —see also [20].

Written in terms of partial derivatives, our system of evolution equations reads

$$\begin{aligned}
\partial_t \phi &= \psi, \\
\partial_t \psi &= -\psi \chi - \frac{d\mathcal{V}}{d\phi}, \\
2\partial_t \chi_{(bd)} - 2e'_{(b}{}^0 \partial_t a_{d)} - 2e'_{(b}{}^j \partial_j a_{d)} &= \frac{2}{3} \left(\mathcal{V}(\phi) - \psi^2 - \frac{1}{3} \chi^2 \right) h_{bd} - (\gamma^p{}_{bd} + \gamma^p{}_{db}) a_p \\
&\quad - \frac{4}{3} \chi (\chi^{ST})_{bd} - 2 (\chi^{ST})_b{}^p (\chi^{ST})_{pd} + 2a_b a_d + 2E_{bd}, \\
\partial_t a_c - e_p{}^0 \partial_t \chi_c{}^p - e_p{}^j \partial_j \chi_c{}^p &= \left(\frac{2}{\psi} \frac{d\mathcal{V}}{d\phi} + \frac{2}{3} \chi \right) a_c - (\chi^{ST})_c{}^b a_b \\
&\quad - \gamma^q{}_{cp} (\chi^{ST})^p{}_q - \gamma^q{}_p{}^p (\chi^{ST})_{cq}, \\
\partial_t E_{bd} - \epsilon^{pa}{}_{(b} e'_{a)}{}^0 \partial_t B_{p|d)} - \epsilon^{pa}{}_{(b} e'_{a)}{}^j \partial_j B_{p|d)} &= -\frac{1}{2} \psi^2 (\chi^{ST})_{bd} - \frac{1}{3} \chi E_{bd} + 5 (\chi^{ST})_{(b}^q E_{d)q} \\
&\quad + 2a_a B_{p(b} \epsilon_{d)}{}^{pa} - \gamma^q{}_{pa} B_{q(d} \epsilon_{b)}{}^{pa} - \epsilon^{pa}{}_{(d} \gamma^q{}_{b)a} B_{pq}, \\
\partial_t B_{bd} - \epsilon^{ap}{}_{(d} e'_{a)}{}^0 \partial_t E_{b)p} - \epsilon^{ap}{}_{(d} e'_{a)}{}^j \partial_j E_{b)p} &= -\frac{1}{3} \chi B_{bd} + 3 (\chi^{ST})_{(b}^q B_{d)q} + (\chi^{ST})_{ac} B_{pq} \epsilon^{pa}{}_{(b} \epsilon_{d)}{}^{qc} \\
&\quad - 2a_a E_{p(b} \epsilon_{d)}{}^{pa} - \gamma^q{}_{pa} E_{q(b} \epsilon_{d)}{}^{ap} - \epsilon^{ap}{}_{(b} \gamma^q{}_{d)a} E_{pq}, \\
\partial_t e_b{}^i &= -\frac{1}{3} \chi e_b{}^i - (\chi^{ST})_b{}^c e_c{}^i, \\
\partial_t e_b{}^0 &= -\frac{1}{3} \chi e_b{}^0 - (\chi^{ST})_b{}^c e_c{}^0 + a_b, \\
\partial_t \gamma'^a{}_{bd} &= -\frac{1}{3} \chi \gamma'^a{}_{bd} + \frac{2}{3} \chi h^{ap} h_{d[p} a_{b]} + B_{dp} \epsilon^{pa}{}_b - (\chi^{ST})_d{}^p \gamma'^a{}_{bp} \\
&\quad + 2h^{ap} (\chi^{ST})_{d[p} a_{b]}.
\end{aligned} \tag{57}$$

This system has clearly the form given by equation (54). If one writes

$$\mathbf{u}^T = (\phi, \psi, \mathbf{z}^T, \mathbf{w}^T, \mathbf{x}^T, \mathbf{y}^T), \tag{58}$$

where

$$\begin{aligned}
\mathbf{z}^T &= (\chi_{11}, \chi_{22}, \chi_{33}, (\chi^{ST})_{12}, (\chi^{ST})_{13}, (\chi^{ST})_{23}, a_1, a_2, a_3), \\
\mathbf{w}^T &= (E_{12}, E_{13}, E_{23}, B_{12}, B_{13}, B_{23}), \\
\mathbf{x}^T &= (e_1^0, e_2^0, e_3^0, e_1^1, e_1^2, e_1^3, e_2^1, e_2^2, e_2^3, e_3^1, e_3^2, e_3^3), \\
\mathbf{y}^T &= (\gamma^1{}_{22}, \gamma^1{}_{33}, \gamma^1{}_{23}, \gamma^2{}_{11}, \gamma^2{}_{33}, \gamma^2{}_{31}, \gamma^3{}_{11}, \gamma^3{}_{22}, \gamma^3{}_{12}),
\end{aligned}$$

then the matrices given in equations (54) and (55) have the explicit form

$$\tilde{\mathbf{A}}^0(\mathbf{u}) = \begin{pmatrix} \mathbf{I}_{2 \times 2} & 0 & 0 & 0 \\ 0 & \tilde{\mathbf{A}}_{9 \times 9}^0 & 0 & 0 \\ 0 & 0 & \tilde{\mathbf{A}}_{6 \times 6}^0 & 0 \\ 0 & 0 & 0 & \mathbf{I}_{21 \times 21} \end{pmatrix}, \quad \mathbf{A}^j(\mathbf{u}) = \begin{pmatrix} \mathbf{0}_{2 \times 2} & 0 & 0 & 0 \\ 0 & \mathbf{A}_{9 \times 9}^j & 0 & 0 \\ 0 & 0 & \mathbf{A}_{6 \times 6}^j & 0 \\ 0 & 0 & 0 & \mathbf{0}_{21 \times 21} \end{pmatrix}, \tag{59}$$

with

$$\tilde{\mathbf{A}}_{9 \times 9}^0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -e_1^0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -e_2^0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -e_3^0 \\ 0 & 0 & 0 & 2 & 0 & 0 & -e_2^0 & -e_1^0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & -e_3^0 & 0 & -e_1^0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & -e_3^0 & -e_2^0 \\ -e_1^0 & 0 & 0 & -e_2^0 & -e_3^0 & 0 & 1 & 0 & 0 \\ 0 & -e_2^0 & 0 & -e_1^0 & 0 & -e_3^0 & 0 & 1 & 0 \\ 0 & 0 & -e_3^0 & 0 & -e_1^0 & -e_2^0 & 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{\mathbf{A}}_{6 \times 6}^0 = \begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{1}{2}e_1^0 & \frac{1}{2}e_2^0 \\ 0 & 1 & 0 & \frac{1}{2}e_1^0 & 0 & -\frac{1}{2}e_3^0 \\ 0 & 0 & 1 & -\frac{1}{2}e_2^0 & \frac{1}{2}e_3^0 & 0 \\ 0 & \frac{1}{2}e_1^0 & -\frac{1}{2}e_2^0 & 1 & 0 & 0 \\ -\frac{1}{2}e_1^0 & 0 & \frac{1}{2}e_3^0 & 0 & 1 & 0 \\ \frac{1}{2}e_2^0 & -\frac{1}{2}e_3^0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{A}_{9 \times 9}^j = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & e_1^j & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_2^j & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_3^j \\ 0 & 0 & 0 & 0 & 0 & 0 & e_2^j & e_1^j & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e_3^j & 0 & e_1^j \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_3^j & e_2^j \\ e_1^j & 0 & 0 & e_2^j & e_3^j & 0 & 0 & 0 & 0 \\ 0 & e_2^j & 0 & e_1^j & 0 & e_3^j & 0 & 0 & 0 \\ 0 & 0 & e_3^j & 0 & e_1^j & e_2^j & 0 & 0 & 0 \end{pmatrix},$$

and

$$\mathbf{A}_{6 \times 6}^j = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{1}{2}e_1^j & -\frac{1}{2}e_2^j \\ 0 & 0 & 0 & -\frac{1}{2}e_1^j & 0 & \frac{1}{2}e_3^j \\ 0 & 0 & 0 & \frac{1}{2}e_2^j & -\frac{1}{2}e_3^j & 0 \\ 0 & -\frac{1}{2}e_1^j & \frac{1}{2}e_2^j & 0 & 0 & 0 \\ \frac{1}{2}e_1^j & 0 & -\frac{1}{2}e_3^j & 0 & 0 & 0 \\ -\frac{1}{2}e_2^j & \frac{1}{2}e_3^j & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It can, therefore, be verified that $\tilde{\mathbf{A}}^0(\mathbf{u})$, $\mathbf{A}^j(\mathbf{u})$ are symmetric and that, furthermore, $\tilde{\mathbf{A}}^0(\mathbf{u})$ is positive definite as long as (56) is satisfied. We summarise the results of this section in the following theorem:

Theorem 1. *The Einstein-Friedrich-nonlinear scalar field (EFsf) system consisting of the equations in (57) forms a quasi-linear first-order symmetric hyperbolic (FOSH) system for the scalar field, its momentum-density, the frame coefficients, the connection coefficients and the electric and magnetic parts of the Weyl tensor, relatively to the slices of constant time t , as long as the quadratic form*

$$\sum_{a=1,2,3} \theta^a \theta^a_j - \frac{D_i \phi}{\psi} \frac{D_j \phi}{\psi},$$

is positive definite.

Using the standard theory of symmetric hyperbolic systems one can then conclude the local existence in time and uniqueness of smooth solutions for the evolution equations implied by the Einstein-nonlinear scalar field. In order to conclude the existence of solutions to the full Einstein-scalar field equations one has to verify that the constraint equations are satisfied during the evolution if these hold initially. This will not be discussed here as the purpose of the present work is to study the evolution of the perturbations. In any case, it follows by general arguments that these are preserved during evolution —see e.g. [20, 19, 7].

Remark. As part of the procedure to make the system (57) explicitly symmetric hyperbolic one has to divide the evolution equation for the acceleration (49) by ψ^2 . This could imply that the system is not well behaved when $\psi = 0$. An inspection shows that the potentially troublesome term is the one containing the first derivative of the potential, which, by virtue of the EFE must be zero in the asymptotic limit. Thus, the coefficient

$$\lim_{t \rightarrow \infty} \frac{\mathcal{V}'}{\psi} = \text{const.} \quad (60)$$

must be required to exist. Moreover, we notice that our gauge restricts the set of solutions, since it only admits scalar fields ϕ which are monotonically decaying functions. Accordingly, the stability of background solutions for which $\phi \rightarrow \infty$ cannot be studied by our method.

5 Stability Analysis

In this section we use the symmetric hyperbolic system derived in last section to show that, for some classes of potentials, the evolution of sufficiently small nonlinear perturbations of a FRW-nonlinear scalar field background prescribed on a Cauchy hypersurface with the topology of a 3-torus \mathbb{T}^3 , either converge to constant values or have an asymptotic exponential decay.

5.1 The background solution

As it is well known, the metric of a Friedman-Robertson-Walker (FRW) spacetime —i.e. an homogeneous and isotropic spacetime— can be written as

$$ds^2 = -dt^2 + \left(\frac{a(t)}{\omega} \right)^2 \delta_{ij} dx^i dx^j,$$

where

$$\omega^2 = 1 + \frac{k}{4} \delta_{ij} x^i x^j, \quad \partial_k \omega^2 = \frac{k}{2} x_k,$$

and the constant $k = -1, 0, 1$ is the curvature of the spatial hypersurfaces. Since the metric is conformal flat, it follows that

$$\mathring{E}_{bd} = \mathring{B}_{bd} = 0.$$

Furthermore, if one foliates the spacetime with the surfaces of constant t , one has that

$$(\mathring{\chi}^{ST})_{bd} = (\mathring{\chi}^A)_{bd} = 0, \quad \mathring{a}_c = 0.$$

For such background metrics

$$\mathring{\chi} = 3\mathring{a}/a \equiv 3H,$$

where $\mathring{\cdot}$ denotes differentiation with respect to time t , and H is the so-called *Hubble function*. In the case of a FRW cosmology the Einstein-scalar field system reduces to the evolution equations

$$\begin{aligned} \frac{d\mathring{\phi}}{dt} &= \mathring{\psi}, \\ \frac{d\mathring{\psi}}{dt} &= -3H\mathring{\psi} - \frac{d\mathcal{V}}{d\mathring{\phi}}, \\ \frac{dH}{dt} &= -H^2 - \frac{1}{3}\mathring{\psi}^2 + \frac{1}{3}\mathcal{V}(\mathring{\phi}), \end{aligned} \tag{61}$$

subject to the Friedmann-scalar field constraint equation

$$H^2(t) - \frac{1}{6}\mathring{\psi}^2(t) - \frac{1}{3}\mathcal{V}(\mathring{\phi}) = -\frac{k}{a^2}. \tag{62}$$

Now, the gauge conditions for the frame are satisfied if one sets

$$\mathring{e}_0^\mu = \delta_0^\mu, \quad \mathring{e}_b^\mu = \left(\frac{\omega}{a} \right) \delta_b^\mu, \quad b = 1, 2, 3,$$

so that the spatial connection coefficients are given by

$$\mathring{\gamma}^c_{bd} = \frac{k}{4a^2} (h_{db}x^c - h_d^c x_b), \quad b, c, d = 1, 2, 3,$$

with $x^\mu = (\omega/a)\delta^\mu_c x^c$. The remaining nonvanishing connection coefficients are

$$\mathring{\gamma}^0_{bd} = Hh_{bd} \quad , \quad \mathring{\gamma}^b_{0d} = Hh^b_d, \quad b, d = 1, 2, 3.$$

5.2 Linearised evolution equations

In this subsection, we derive the linearised system corresponding to the nonlinear equations of Theorem 1, for a FRW background with a selfinteracting scalar field. In order to perform the linearisation procedure we compute

$$\left. \frac{d\check{\mathbf{u}}^\epsilon}{d\epsilon} \right|_{\epsilon=0}$$

and drop all (nonlinear) terms of coupled perturbations. In this way, we obtain the following system

$$\begin{aligned} \partial_t \check{\phi} &= \check{\psi}, \\ \partial_t \check{\psi} &= - \left(\frac{d^2 \check{\mathcal{V}}}{d\check{\phi}^2} \right) \check{\phi} - 3H\check{\psi} - \check{\psi}\check{\chi}, \\ 2\partial_t \check{\chi}_{(bd)} - 2 \left(\frac{\omega}{a} \right) \delta_{(d}{}^j \partial_j \check{a}_{b)} &= \frac{2}{3} \left(\left(\frac{d\check{\mathcal{V}}}{d\check{\phi}} \right) \check{\phi} - 2\check{\psi}\check{\psi} - 2H\check{\chi} \right) h_{bd} - 4H(\check{\chi}^{ST})_{bd} \\ &\quad - (\check{\gamma}^p{}_{bd} + \check{\gamma}^p{}_{db}) \check{a}_p + 2\check{E}_{bd}, \\ \partial_t \check{E}_{bd} - \left(\frac{\omega}{a} \right) \epsilon^{pa}{}_{(b} \delta_a{}^j \partial_j \check{B}_{p|d)} &= -\frac{1}{2} \check{\psi}^2 (\check{\chi}^{ST})_{bd} - H\check{E}_{bd} - \check{\gamma}^q{}_{pa} \check{B}_{q(d} \epsilon_{b)}{}^{pa} - \epsilon^{pa}{}_{(d} \check{\gamma}^q{}_{|b)a} \check{B}_{pq}, \\ \partial_t \check{B}_{bd} - \left(\frac{\omega}{a} \right) \epsilon_{(d}{}^{ap} \delta_a{}^j \partial_j \check{E}_{p|b)} &= -\check{\gamma}^q{}_{pa} \check{E}_{q(b} \epsilon_{d)}{}^{ap} - \epsilon^{ap}{}_{(b} \check{\gamma}^q{}_{d)a} \check{E}_{pq} - H\check{B}_{bd}, \\ \partial_t \check{a}_c - \left(\frac{\omega}{a} \right) \delta_p{}^j \partial_j \check{\chi}_c{}^p &= -\check{\gamma}^q{}_{cp} (\check{\chi}^{ST})_q{}^p - \check{\gamma}^q{}_p{}^p (\check{\chi}^{ST})_{cq} + \left(2H + \frac{2}{\check{\psi}} \frac{d\check{\mathcal{V}}}{d\check{\phi}} \right) \check{a}_c + \left(\frac{dH}{dt} \right) \check{e}_c{}^0, \\ \partial_t \check{e}_b{}^0 &= -H\check{e}_b{}^0 + \check{a}_b, \\ \partial_t \check{e}_b{}^i &= -\frac{1}{3} \left(\frac{\omega}{a} \right) \delta_b{}^i \check{\chi} - \left(\frac{\omega}{a} \right) \delta_c{}^i (\check{\chi}^{ST})_b{}^c - H\check{e}_b{}^i, \\ \partial_t \check{\gamma}^a{}_{bd} &= -\frac{1}{3} \check{\gamma}^a{}_{bd} \check{\chi} - \check{\gamma}^a{}_{bp} (\check{\chi}^{ST})_d{}^p + H(\delta_d^a \check{a}_b - h_{bd} \check{a}^a) \\ &\quad + \check{B}_{dp} \epsilon^{pa}{}_b - H\check{\gamma}^a{}_{bd}. \end{aligned} \tag{63}$$

As a consequence, the linearised system has the following form

$$\mathring{\mathbf{A}}^0 \partial_t \check{\mathbf{u}} - \mathring{\mathbf{A}}^j(t, \mathbf{x}) \partial_j \check{\mathbf{u}} = \mathring{\mathbf{B}}(t, \mathbf{x}) \check{\mathbf{u}}. \tag{64}$$

If one defines $\check{\mathbf{u}}$ in the same way as in equation (58) one obtains the linearised matrix $\mathring{\mathbf{B}}(t, \mathbf{x})$ given by

$$\begin{pmatrix} \mathbf{B}_{2 \times 2}^{(1)} & \mathbf{B}_{2 \times 3}^{(2)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{B}_{3 \times 2}^{(3)} & \mathbf{B}_{3 \times 3}^{(4)} & 0 & -\frac{k}{4a\omega} \mathbf{B}_{3 \times 3}^{(5)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4H\mathbf{I}_{3 \times 3} & \frac{k}{4a\omega} \mathbf{B}_{3 \times 3}^{(6)} & 2\mathbf{I}_{3 \times 3} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{3k}{4a\omega} \mathbf{B}_{3 \times 3}^{(7)} & \left(\frac{2}{\check{\psi}} \check{\mathcal{V}}' + 2H \right) \mathbf{I}_{3 \times 3} & 0 & 0 & \frac{dH}{dt} \mathbf{I}_{3 \times 3} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \check{\psi}^2 \mathbf{I}_{3 \times 3} & 0 & -H\mathbf{I}_{3 \times 3} & \frac{k}{8a\omega} \mathbf{B}_{3 \times 3}^{(8)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{k}{8a\omega} \mathbf{B}_{3 \times 3}^{(9)} & -H\mathbf{I}_{3 \times 3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{I}_{3 \times 3} & 0 & 0 & -H\mathbf{I}_{3 \times 3} & 0 & 0 \\ 0 & -\frac{\omega}{3a} \mathbf{B}_{9 \times 3}^{(10)} & -\frac{\omega}{a} \mathbf{B}_{9 \times 3}^{(11)} & 0 & 0 & 0 & 0 & -H\mathbf{I}_{9 \times 9} & 0 \\ 0 & -\frac{k}{12a\omega} \mathbf{B}_{9 \times 3}^{(12)} & \frac{k}{4a\omega} \mathbf{B}_{9 \times 3}^{(13)} & -H\mathbf{B}_{9 \times 3}^{(14)} & 0 & \mathbf{B}_{9 \times 3}^{(15)} & 0 & 0 & -H\mathbf{I}_{9 \times 9} \end{pmatrix}$$

where

$$\mathbf{B}_{2 \times 2}^{(1)} = \begin{pmatrix} 0 & 1 \\ -\check{\mathcal{V}}'' & -3H \end{pmatrix}, \quad \mathbf{B}_{2 \times 3}^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ -\check{\psi} & -\check{\psi} & -\check{\psi} \end{pmatrix}, \quad \mathbf{B}_{3 \times 2}^{(3)} = \begin{pmatrix} \frac{1}{3} \check{\mathcal{V}}' & -\frac{2}{3} \check{\psi} \\ \frac{1}{3} \check{\mathcal{V}}' & -\frac{2}{3} \check{\psi} \\ \frac{1}{3} \check{\mathcal{V}}' & -\frac{2}{3} \check{\psi} \end{pmatrix},$$

$$\mathbf{B}_{3 \times 3}^{(4)} = \begin{pmatrix} -\frac{2}{3} H & -\frac{2}{3} H & -\frac{2}{3} H \\ -\frac{2}{3} H & -\frac{2}{3} H & -\frac{2}{3} H \\ -\frac{2}{3} H & -\frac{2}{3} H & -\frac{2}{3} H \end{pmatrix}, \quad \mathbf{B}_{3 \times 3}^{(5)} = \begin{pmatrix} 0 & x^j \delta_j^2 & x^j \delta_j^3 \\ x^j \delta_j^1 & 0 & x^j \delta_j^3 \\ x^j \delta_j^1 & x^j \delta_j^2 & 0 \end{pmatrix}, \quad \mathbf{B}_{3 \times 3}^{(6)} = \begin{pmatrix} x_j \delta_2^j & x_j \delta_1^j & 0 \\ x_j \delta_3^j & 0 & x_j \delta_1^j \\ 0 & x_j \delta_3^j & x_j \delta_2^j \end{pmatrix},$$

$$\begin{pmatrix} x^j \delta_j^2 & x^j \delta_j^3 & 0 \\ x^j \delta_j^1 & 0 & x^j \delta_j^3 \\ 0 & x^j \delta_j^1 & x^j \delta_j^2 \end{pmatrix}, \quad \mathbf{B}_{3 \times 3}^{(8)} = \begin{pmatrix} 0 & -x^j \delta_j^1 & x^j \delta_j^2 \\ x^j \delta_j^1 & 0 & -x^j \delta_j^3 \\ -x^j \delta_j^2 & x^j \delta_j^3 & 0 \end{pmatrix}, \quad \mathbf{B}_{3 \times 3}^{(9)} = \begin{pmatrix} 0 & x^j \delta_j^1 & -x^j \delta_j^2 \\ -x^j \delta_j^1 & 0 & x^j \delta_j^3 \\ x^j \delta_j^2 & -x^j \delta_j^3 & 0 \end{pmatrix},$$

$$\mathbf{B}_{9 \times 3}^{(10)} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad \mathbf{B}_{9 \times 3}^{(11)} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{B}_{9 \times 3}^{(12)} = \begin{pmatrix} \delta_j^1 x^j & \delta_j^1 x^j & \delta_j^1 x^j \\ \delta_j^1 x^j & \delta_j^1 x^j & \delta_j^1 x^j \\ 0 & 0 & 0 \\ \delta_j^2 x^j & \delta_j^2 x^j & \delta_j^2 x^j \\ \delta_j^2 x^j & \delta_j^2 x^j & \delta_j^2 x^j \\ 0 & 0 & 0 \\ \delta_j^3 x^j & \delta_j^3 x^j & \delta_j^3 x^j \\ \delta_j^3 x^j & \delta_j^3 x^j & \delta_j^3 x^j \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{B}_{9 \times 3}^{(13)} = \begin{pmatrix} \delta_2^j x_j & 0 & 0 \\ 0 & \delta_3^j x_j & 0 \\ 0 & \delta_2^j x_j & -\delta_j^1 x^j \\ \delta_1^j x_j & 0 & 0 \\ 0 & 0 & \delta_3^j x_j \\ \delta_3^j x_j & -\delta_j^2 x^j & 0 \\ 0 & \delta_1^j x_j & 0 \\ 0 & 0 & \delta_2^j x_j \\ -\delta_j^3 x^j & 0 & \delta_1^j x_j \end{pmatrix}, \quad \mathbf{B}_{9 \times 3}^{(14)} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{B}_{9 \times 3}^{(15)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This matrix $\mathring{\mathbf{B}}$ is made of blocks around the diagonal and is close to being of lower triangular form. In fact, it could easily be cast into that form by ‘removing’ the upper matrix $\mathring{\mathbf{B}}^{(5)}$. In the more general case where $k \neq 0$ one has three main blocks. A first block consists of the matrices $\mathring{\mathbf{B}}^{(1)}, \mathring{\mathbf{B}}^{(2)}, \mathring{\mathbf{B}}^{(3)}, \mathring{\mathbf{B}}^{(4)}$, related to the perturbation variables $(\check{\phi}, \check{\psi}, \check{\chi}_{11}, \check{\chi}_{22}, \check{\chi}_{33})$. A second block is related to the variables $((\check{\chi}^{ST})_{bd}, \check{a}_c, \check{E}_{bd}, \check{B}_{bd}, \check{e}_b^{\prime 0})$ and a third one for the perturbation of the spatial connection and the remaining spatial frame coefficients. This last block does not play an important role in the stability results.

5.3 Results on the nonlinear stability of symmetric hyperbolic systems

In this subsection we discuss the results of the theory of symmetric hyperbolic systems required to discuss the nonlinear stability of perturbations of a reference solution to *the Einstein-scalar field system with flat spatial sections* —i.e. $k = 0$. The analysis carried out in the previous sections shows that the Cauchy problem for these nonlinear perturbations takes the form

$$\begin{aligned} & \left(\mathring{\mathbf{A}}^0 - \epsilon \mathring{\mathbf{A}}^0(t, \check{\mathbf{u}}, \epsilon) \right) \partial_t \check{\mathbf{u}} - \left(\mathring{\mathbf{A}}^j(t) + \epsilon \mathring{\mathbf{A}}^j(t, \check{\mathbf{u}}, \epsilon) \right) \partial_j \check{\mathbf{u}} = \left(\mathring{\mathbf{B}}(t) + \epsilon \mathring{\mathbf{B}}(t, \check{\mathbf{u}}, \epsilon) \right) \check{\mathbf{u}}, \\ & \check{\mathbf{u}}(\mathbf{x}, 0) = \check{\mathbf{u}}_0(\mathbf{x}), \end{aligned} \tag{65}$$

where $\mathring{\mathbf{A}}^0$ is a constant diagonal matrix with positive entries, while $\mathring{\mathbf{A}}^j$ and $\mathring{\mathbf{B}}$ are matrices whose entries are smooth functions of the time coordinate t . The matrices $\mathring{\mathbf{A}}^j$, $j = 1, 2, 3$, are symmetric. Furthermore, $\mathring{\mathbf{A}}^0$, $\mathring{\mathbf{A}}^j$ and $\mathring{\mathbf{B}}$ vanish at least linearly with $\check{\mathbf{u}}$. If $k = 0$, the linearised system (64) reduces to

$$\mathring{\mathbf{A}}^0 \partial_t \check{\mathbf{u}} - \mathring{\mathbf{A}}^j(t) \partial_j \check{\mathbf{u}} = \mathring{\mathbf{B}}(t) \check{\mathbf{u}}.$$

Our analysis will require the following definition:

Definition 1 (Relaxed stability eigenvalue condition). *A system of the form given by (65) is said to satisfy the relaxed stability eigenvalue condition if the following conditions hold:*

(i) There is a constant δ_0 with $0 < \delta_0 \leq \delta(t)$ such that, the eigenvalues $\lambda(t, \xi)$ of the symbol

$$i\xi_j \mathring{\mathbf{A}}^j(t) + \mathring{\mathbf{B}}(t), \quad i = \sqrt{-1}.$$

satisfy

$$\operatorname{Re}(\lambda(t, \xi)) \leq -\delta(t)$$

for all n -tuples of integers $\xi = (\xi_1, \dots, \xi_n) \neq 0$.

(ii) There is a constant δ_0 with $0 < \delta_0 \leq \delta(t)$ such that, for $\xi = \mathbf{0}$, the eigenvalues $\lambda(t, \mathbf{0})$ of $\mathring{\mathbf{B}}(t)$, satisfy either

$$\operatorname{Re}(\lambda(t, \mathbf{0})) \leq -\delta(t) \quad \text{or} \quad \lambda(t, \mathbf{0}) = 0.$$

If the multiplicity of the vanishing eigenvalue is m , then there are m linearly independent eigenvectors.

(iii) The Kernel of $\check{\mathbf{B}}$ contains the Kernel of $\mathring{\mathbf{B}}$.

In order to state the result that allows us to ascertain the nonlinear stability for the solutions of the Cauchy problem (65), we introduce some notation. Let $H^k(\mathbb{T}^n; \mathbb{R}^s)$ be the space of all summable functions $\mathbf{u}(\cdot, t) : \mathbb{T}^n \rightarrow \mathbb{R}^s$ such that for each multi-index $|\alpha| \leq k$, $D^\alpha \mathbf{u}(\mathbf{x}, t)$ exists in the weak sense and belongs to $L^2(\mathbb{T}^n)$. The norm in $H^k(\mathbb{T}^n; \mathbb{R}^s)$ is defined by

$$\|\mathbf{u}(t)\|_{H^k(\mathbb{T}^n)} \equiv \left(\sum_{|\alpha|=0}^k \int_{\mathbb{T}^n} |D^\alpha \mathbf{u}|^2 d\mathbf{x} \right)^{1/2},$$

where

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}.$$

Given a perturbation $\check{\mathbf{u}}(\mathbf{x}, t)$, we denote by $\hat{\mathbf{u}}(\xi, t)$ the Fourier coefficient corresponding to the frequency vector ξ —recall that it is assumed that the topology of the spatial sections is taken to be that of \mathbb{T}^3 . In particular, $\hat{\mathbf{u}}(\mathbf{0}, t)$ denotes the coefficient corresponding to $\xi = \mathbf{0}$.

The following theorem shows the exponential decay or convergence to constant values of a solution to the nonlinear system (65). Its proof follows closely the methods of [9, 21] and will be omitted for the sake of brevity.

Theorem 2. *Let \mathbf{P} denote the projector into the Kernel of $\mathring{\mathbf{B}}(t)$. Furthermore, let $\check{\mathbf{u}}^{(0)} \equiv \mathbf{P}\hat{\mathbf{u}}(\mathbf{0}, t)$ and $\check{\mathbf{v}} \equiv \check{\mathbf{u}} - \check{\mathbf{u}}^{(0)}$. If the symmetric hyperbolic system of the Cauchy problem (65) satisfies the relaxed stability eigenvalue condition, then the system is stable in the sense that, for sufficiently small ϵ there is $\epsilon_0 \geq \epsilon > 0$ such that*

$$\lim_{t \rightarrow \infty} \|\check{\mathbf{v}}(t)\|_{H^{n+2}(\mathbb{T}^n; \mathbb{R}^s)} = 0$$

and $\check{\mathbf{u}}^{(0)}(t) \rightarrow \text{const.}$ as time goes to infinity.

Remark. The use of Fourier methods in Theorem 2 does not allow to disentangle which fields (or combination thereof) decay to a constant. However, it is possible to identify the blocks in $\mathring{\mathbf{B}}(t)$ responsible for this behaviour.

In the remainder of the article, it will be analysed under which conditions the symmetric hyperbolic evolution equations for the perturbations of a reference solution to the Einstein-scalar field system satisfy the requirements of the relaxed stability eigenvalue condition (Definition 1) so that Theorem 2 can be used.

5.4 Satisfying Assumption (ii) of Definition 1

In the case of a flat background ($k = 0$, $\omega = 1$), the linearized matrices $\mathring{\mathbf{A}}$ and $\mathring{\mathbf{B}}$ are functions of time only. Because of the block structure of the matrix, the characteristic polynomial of $\mathring{\mathbf{B}}$ is the product of the characteristic polynomials of the block matrices around the diagonal. A computation gives the following polynomial:

$$(\lambda + H)^{21} \times \left(\lambda^2 - \left(2\frac{\mathring{\mathcal{V}}'}{\mathring{\psi}} + H \right) \lambda - \frac{dH}{dt} - 2H^2 - 2H\frac{\mathring{\mathcal{V}}'}{\mathring{\psi}} \right)^3 \times \left(\lambda^2 + 5H\lambda + \mathring{\psi}^2 + 4H^2 \right)^3 \times \left(\lambda^5 + 5H\lambda^4 - \left(2\mathring{\psi}^2 - 6H^2 - \mathring{\mathcal{V}}'' \right) \lambda^3 + \left(2H\mathring{\mathcal{V}}'' + \mathring{\psi}\mathring{\mathcal{V}}' \right) \lambda^2 \right). \quad (66)$$

The first term in the expression (66) is the characteristic polynomial of the block related with the perturbation variables $(\mathring{B}_{bd}, \mathring{\epsilon}_b'^j, \mathring{\gamma}_{ab}^c)$. The second term is related to the variables $(\mathring{a}_c, \mathring{\epsilon}_b'^0)$, and the third one to $((\mathring{\chi}^{ST})_{bd}, \mathring{E}_{bd})$. Finally, the last term —the fifth order polynomial— arises from the block of unknowns $(\mathring{\phi}, \mathring{\psi}, \mathring{\chi}_{pp})$.

In order to obtain conditions from the characteristic polynomial we will make use of results from the *Liénard-Chipart theorem*. The latter gives necessary and sufficient conditions for a polynomial with real coefficients to have roots with negative real part —see e.g. [23]. This type of polynomials are called *Hurwitz polynomials*.

Theorem 3 (Liénard-Chipart). *Let*

$$f(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_n, \quad a_0 > 0,$$

be a polynomial with real coefficients. Then the following statements are equivalent:

- (i) *the polynomial is a Hurwitz polynomial;*
- (ii) *The coefficients of f are positive and $\delta_2 > 0$, $\delta_4 > 0, \dots, \delta_n$, n even;*
- (iii) *The coefficients of f are positive and $\delta_1 > 0$, $\delta_3 > 0, \dots, \delta_n$, n odd,*

where the Hurwitz determinants are defined by

$$\delta_0 \equiv 1, \quad \delta_l \equiv \det \begin{pmatrix} a_1 & a_3 & a_5 & \dots & a_{2l-1} \\ 1 & a_2 & a_4 & \dots & a_{2l-2} \\ 0 & a_1 & a_3 & \dots & a_{2l-3} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & a_l \end{pmatrix} \quad l = 1, \dots, n.$$

It is easy to see that the first term in (66) requires

$$H(t) > 0 \quad (67)$$

which is simply the condition for an (ever) expanding background. From the second factor in the characteristic polynomial (66), one obtains the following conditions on $\mathring{\mathcal{V}}'$:

$$\frac{\mathring{\mathcal{V}}'}{\mathring{\psi}} < -\frac{H}{2}, \quad 2H\frac{\mathring{\mathcal{V}}'}{\mathring{\psi}} < -\frac{dH}{dt} - 2H^2. \quad (68)$$

Notice that by construction we have that $\psi < 0$ —see equation (39). The first condition in (68) implies that the first derivative of the scalar field potential is positive. The second condition is related to the evolution equation for the Hubble function (61). Moreover, by virtue of condition (60) the ratio $\mathring{\mathcal{V}}'/\mathring{\psi}$ tends to a constant which is bigger than the asymptotic value of $H/2$.

The fifth order polynomial in (66) can be rewritten as

$$\lambda^2 \left(\lambda^3 + 5H\lambda^2 - \left(2\mathring{\psi}^2 - 6H^2 - \mathring{\mathcal{V}}'' \right) \lambda + \left(2H\mathring{\mathcal{V}}'' + \mathring{\psi}\mathring{\mathcal{V}}' \right) \right)$$

from where one sees that the matrix $\mathring{\mathbf{B}}(t)$ has a double vanishing eigenvalue. A direct application of the Liénard-Chipart Theorem to the remaining third order polynomial renders the following two conditions:

$$\mathring{\mathcal{V}}'' > -\mathring{\psi} \frac{\mathring{\mathcal{V}}'}{2H}, \quad \mathring{\mathcal{V}}'' - \mathring{\psi} \frac{\mathring{\mathcal{V}}'}{3H} > 10 \left(-H^2 + \frac{\mathring{\psi}^2}{3} \right). \quad (69)$$

The first condition relates the second derivative of the potential to its first derivative, and implies that $\mathring{\mathcal{V}}''$ has to be positive and bounded. The second condition is sharper and also relates to the evolution of the Hubble function (61). Taking the limit $\psi \rightarrow 0$, one recovers the necessary asymptotic conditions for stability found in [11].

5.4.1 Nonflat backgrounds

For the sake of conciseness, the case of a reference solution with nonflat spatial sections will not be treated here. Nevertheless, we point out some features of the required analysis. For this class of solutions the characteristic polynomial of the matrix $\mathring{\mathbf{B}}$ is given by

$$f(\lambda) \times \left(\lambda^5 + 5H\lambda^4 - \left(2\mathring{\psi}^2 - 6H^2 - \mathring{\mathcal{V}}'' \right) \lambda^3 + \left(2H\mathring{\mathcal{V}}'' + \mathring{\psi}\mathring{\mathcal{V}}' \right) \lambda^2 \right) \times (\lambda + H)^{18}, \quad (70)$$

where $f(\lambda)$ is a polynomial of degree 15 in λ . The coefficients of this polynomial are, unfortunately, large expressions without an obvious structure. Accordingly, an analysis of the roots by means of the Liénard-Chipart Theorem, although in principle possible, becomes extremely cumbersome. The study of perturbations of cosmological models with nonflat spatial sections also requires to take into account extra terms coming from partial differentiation when constructing the Sobolev norms. This is because the matrices $\mathring{\mathbf{A}}^j$ and $\mathring{\mathbf{B}}$ are now functions of the coordinates x^i .

Finally, it is mentioned that a discussion of the perturbations of reference solutions with nonflat spatial sections requires suitable adaptations of the relaxed stability eigenvalue condition (Definition 1) and of Theorem 2.

5.5 Satisfying Assumption (i) of Definition 1

The purpose of the present section is to show that Assumption (i) of the *stability eigenvalue condition* follows from Assumption (ii). This turns out to be a consequence of a theorem due to Ortiz in [21] —see also [10].

Theorem 4. *Let \mathbf{S} be a nonsingular matrix which takes the $s \times s$ matrix $\mathring{\mathbf{B}}$ into block diagonal form. If for all unit frequency vectors $\boldsymbol{\xi}$, the linear map*

$$\mathbf{i}\mathbf{S}^{-1}\mathring{\mathbf{A}}^0\mathbf{S} + \mathbf{i}\mathring{\mathbf{A}}(t, \boldsymbol{\xi}) : \text{Ker}(\mathring{\mathbf{B}}(t)) \rightarrow \mathbb{C}^s$$

with

$$\mathring{\mathbf{A}}(t, \boldsymbol{\xi}) \equiv -\mathbf{i}\mathbf{S}^{-1}\boldsymbol{\xi}_j\mathring{\mathbf{A}}^j\mathbf{S},$$

is injective, then there are positive constants ξ_0 , δ_0 and $\mu_0 = \delta_0/c$, such that all eigenvalues $\lambda(t, \boldsymbol{\xi})$ of the symbol $\mathbf{i}\boldsymbol{\xi}_j\mathring{\mathbf{A}}^j(t) + \mathring{\mathbf{B}}(t)$ satisfy

$$\text{Re}\lambda(t, \boldsymbol{\xi}) \leq \begin{cases} -\mu_0|\boldsymbol{\xi}|^2 \text{ or } -\delta_0, & \text{if } |\boldsymbol{\xi}| \leq \xi_0 \\ -\delta_0, & \text{if } |\boldsymbol{\xi}| \geq \xi_0 \end{cases}.$$

Remark. This theorem states that if the determinant of the linear map is different from zero, then the eigenvalues of the linearised symbol have negative real part for nonzero Fourier frequency, decaying no faster than $|\boldsymbol{\xi}|^2$ when $|\boldsymbol{\xi}| \rightarrow 0$.

In order to apply this theorem, we must first put $\mathring{\mathbf{B}}$ into block diagonal form. To this end we need to find a transformation, \mathbf{S} , such that

$$\mathbf{S}^{-1}\mathring{\mathbf{B}}\mathbf{S} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathring{\mathbf{B}} \end{pmatrix}. \quad (71)$$

To achieve this, it is sufficient to consider the block matrix corresponding to the degree 5 polynomial which contains the two zero eigenvalues. Let us define the 5×5 matrix $\mathring{\mathbf{M}}$ by

$$\mathring{\mathbf{M}} = \begin{pmatrix} \mathbf{B}_{2 \times 2}^{(1)} & \mathbf{B}_{2 \times 3}^{(2)} \\ \mathbf{B}_{3 \times 2}^{(3)} & \mathbf{B}_{3 \times 3}^{(4)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -\mathring{\mathcal{V}}'' & -3H & -\mathring{\psi} & -\mathring{\psi} & -\mathring{\psi} \\ \frac{1}{3}\mathring{\mathcal{V}}' & -\frac{2}{3}\mathring{\psi} & -\frac{2}{3}H & -\frac{2}{3}H & -\frac{2}{3}H \\ \frac{1}{3}\mathring{\mathcal{V}}' & -\frac{2}{3}\mathring{\psi} & -\frac{2}{3}H & -\frac{2}{3}H & -\frac{2}{3}H \\ \frac{1}{3}\mathring{\mathcal{V}}' & -\frac{2}{3}\mathring{\psi} & -\frac{2}{3}H & -\frac{2}{3}H & -\frac{2}{3}H \end{pmatrix}.$$

Once $\mathring{\mathbf{M}}$ has put into block diagonal form, it turns out that the condition (71) holds. In order to find a suitable transformation, $\mathring{\mathbf{S}}$, we first make a change of basis taking $\mathring{\mathbf{M}}$ into block triangular form. For this, we compute the eigenvectors corresponding to the two vanishing eigenvalues. From the condition

$$\mathring{\mathbf{M}}\mathbf{v} = \mathbf{0},$$

it follows that the eigenvectors can be taken to be

$$\mathbf{v}_1 = \begin{pmatrix} -\frac{3\mathring{\psi}}{\mathring{\mathcal{V}}''} \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} \frac{6H}{\mathring{\mathcal{V}}'} \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

With the help of these two eigenvectors we can easily transform $\mathring{\mathbf{M}}$ into block triangular form by completing the basis with three further linearly independent eigenvectors. For instance, one can consider

$$\mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

Using these basis vectors as the columns of a transformation matrix one arrives to

$$\mathbf{P} \equiv \begin{pmatrix} -\frac{3\mathring{\psi}}{\mathring{\mathcal{V}}''} & \frac{6H}{\mathring{\mathcal{V}}'} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix}$$

with inverse given by

$$\mathbf{P}^{-1} = \begin{pmatrix} -\frac{1}{3} \frac{\mathring{\mathcal{V}}'' \mathring{\mathcal{V}}'}{2H\mathring{\mathcal{V}}'' + \mathring{\psi}\mathring{\mathcal{V}}'} & 0 & \frac{2H\mathring{\mathcal{V}}''}{2H\mathring{\mathcal{V}}'' + \mathring{\psi}\mathring{\mathcal{V}}'} & \frac{2H\mathring{\mathcal{V}}''}{2H\mathring{\mathcal{V}}'' + \mathring{\psi}\mathring{\mathcal{V}}'} & -\frac{2H\mathring{\mathcal{V}}''}{2H\mathring{\mathcal{V}}'' + \mathring{\psi}\mathring{\mathcal{V}}'} \\ \frac{1}{3} \frac{\mathring{\mathcal{V}}'' \mathring{\mathcal{V}}'}{2H\mathring{\mathcal{V}}'' + \mathring{\psi}\mathring{\mathcal{V}}'} & 0 & \frac{\mathring{\psi}\mathring{\mathcal{V}}'}{2H\mathring{\mathcal{V}}'' + \mathring{\psi}\mathring{\mathcal{V}}'} & \frac{\mathring{\psi}\mathring{\mathcal{V}}'}{2H\mathring{\mathcal{V}}'' + \mathring{\psi}\mathring{\mathcal{V}}'} & -\frac{\mathring{\psi}\mathring{\mathcal{V}}'}{2H\mathring{\mathcal{V}}'' + \mathring{\psi}\mathring{\mathcal{V}}'} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & 1 \end{pmatrix}.$$

As a result of these computations one finds that

$$\mathbf{P}^{-1} \mathring{\mathbf{M}} \mathbf{P} = \begin{pmatrix} \mathbf{L}_{3 \times 3} & \mathbf{H}_{3 \times 2} \\ \mathbf{0}_{2 \times 3} & \mathbf{0}_{2 \times 2} \end{pmatrix}$$

with

$$\mathbf{L}_{3 \times 3} \equiv \begin{pmatrix} -2H & 0 & -\frac{\mathring{\mathcal{V}}''}{3} \frac{4H\mathring{\psi} + \mathring{\mathcal{V}}'}{2H\mathring{\mathcal{V}}'' + \mathring{\psi}\mathring{\mathcal{V}}'} \\ -\mathring{\psi} \frac{\mathring{\mathcal{V}}'}{\mathring{\mathcal{V}}''} & 0 & -\frac{\mathring{\mathcal{V}}'}{3} \frac{2\mathring{\psi}^2 - \mathring{\mathcal{V}}''}{2H\mathring{\mathcal{V}}'' + \mathring{\psi}\mathring{\mathcal{V}}'} \\ 0 & -\frac{3}{\mathring{\mathcal{V}}'} (2H\mathring{\mathcal{V}}'' + \mathring{\psi}\mathring{\mathcal{V}}') & -3H \end{pmatrix}, \quad \mathbf{H}_{3 \times 2} \equiv \begin{pmatrix} -\frac{8}{3} \frac{H^2 \mathring{\mathcal{V}}''}{2H\mathring{\mathcal{V}}'' + \mathring{\psi}\mathring{\mathcal{V}}'} & -\frac{8}{3} \frac{H^2 \mathring{\mathcal{V}}''}{2H\mathring{\mathcal{V}}'' + \mathring{\psi}\mathring{\mathcal{V}}'} \\ -\frac{4}{3} \frac{H\mathring{\psi}\mathring{\mathcal{V}}'}{2H\mathring{\mathcal{V}}'' + \mathring{\psi}\mathring{\mathcal{V}}'} & -\frac{4}{3} \frac{H\mathring{\psi}\mathring{\mathcal{V}}'}{2H\mathring{\mathcal{V}}'' + \mathring{\psi}\mathring{\mathcal{V}}'} \\ -2\mathring{\psi} & -2\mathring{\psi} \end{pmatrix}.$$

It can be readily verified that the matrix \mathbf{L} is negative definite as long as the conditions (69) of the previous section hold. Now, in order to proceed further we need another transformation which converts the block upper triangular form into block diagonal. Choosing

$$\mathbf{R} \equiv \begin{pmatrix} \mathbf{I}_{3 \times 3} & \mathbf{Q}_{3 \times 2} \\ \mathbf{0}_{2 \times 3} & \mathbf{I}_{2 \times 2} \end{pmatrix}, \quad \mathbf{R}^{-1} = \begin{pmatrix} \mathbf{I}_{3 \times 3} & -\mathbf{Q}_{3 \times 2} \\ \mathbf{0}_{2 \times 3} & \mathbf{I}_{2 \times 2} \end{pmatrix}$$

one finds that

$$\mathbf{R}^{-1} (\mathbf{P}^{-1} \mathring{\mathbf{M}} \mathbf{P}) \mathbf{R} = \begin{pmatrix} \mathbf{L} & \mathbf{LQ} + \mathbf{H} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Thus, in order to obtain the required block-diagonal form we just need to solve

$$\mathbf{LQ} = -\mathbf{H}.$$

A computation shows that

$$\mathbf{Q} = \begin{pmatrix} -\frac{4}{3} \frac{H\dot{\mathcal{V}}''}{2H\dot{\mathcal{V}}'' + \dot{\psi}\dot{\mathcal{V}}'} & -\frac{4}{3} \frac{H\dot{\mathcal{V}}''}{2H\dot{\mathcal{V}}'' + \dot{\psi}\dot{\mathcal{V}}'} \\ -\frac{2}{3} \frac{\dot{\psi}\dot{\mathcal{V}}'}{2H\dot{\mathcal{V}}'' + \dot{\psi}\dot{\mathcal{V}}'} & -\frac{2}{3} \frac{\dot{\psi}\dot{\mathcal{V}}'}{2H\dot{\mathcal{V}}'' + \dot{\psi}\dot{\mathcal{V}}'} \\ 0 & 0 \end{pmatrix},$$

so that the whole transformation matrix \mathbf{S} is given by

$$\mathbf{S} \equiv \begin{pmatrix} (\mathbf{PR})_{5 \times 5} & \mathbf{0}_{5 \times 30} \\ \mathbf{0}_{30 \times 5} & \mathbf{I}_{30 \times 30} \end{pmatrix}$$

where

$$\mathbf{PR} = \begin{pmatrix} -\frac{3\dot{\psi}}{\dot{\mathcal{V}}''} & \frac{6H}{\dot{\mathcal{V}}'} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & \frac{1}{3} & -\frac{2}{3} \\ 1 & 1 & 0 & -\frac{2}{3} & \frac{1}{3} \\ 1 & 1 & 0 & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

with inverse given by

$$(\mathbf{PR})^{-1} = \begin{pmatrix} -\frac{1}{3} \frac{\dot{\mathcal{V}}'\dot{\mathcal{V}}''}{2H\dot{\mathcal{V}}'' + \dot{\psi}\dot{\mathcal{V}}'} & 0 & \frac{2}{3} \frac{H\dot{\mathcal{V}}''}{2H\dot{\mathcal{V}}'' + \dot{\psi}\dot{\mathcal{V}}'} & \frac{2}{3} \frac{H\dot{\mathcal{V}}''}{2H\dot{\mathcal{V}}'' + \dot{\psi}\dot{\mathcal{V}}'} & \frac{2}{3} \frac{H\dot{\mathcal{V}}''}{2H\dot{\mathcal{V}}'' + \dot{\psi}\dot{\mathcal{V}}'} \\ \frac{1}{3} \frac{\dot{\mathcal{V}}'\dot{\mathcal{V}}''}{2H\dot{\mathcal{V}}'' + \dot{\psi}\dot{\mathcal{V}}'} & 0 & \frac{1}{3} \frac{\dot{\psi}\dot{\mathcal{V}}'}{2H\dot{\mathcal{V}}'' + \dot{\psi}\dot{\mathcal{V}}'} & \frac{1}{3} \frac{\dot{\psi}\dot{\mathcal{V}}'}{2H\dot{\mathcal{V}}'' + \dot{\psi}\dot{\mathcal{V}}'} & \frac{1}{3} \frac{\dot{\psi}\dot{\mathcal{V}}'}{2H\dot{\mathcal{V}}'' + \dot{\psi}\dot{\mathcal{V}}'} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & 1 \end{pmatrix}.$$

Summarising: we have constructed the matrix \mathbf{S} which transforms the linearised matrix $\mathring{\mathbf{B}}$ into block diagonal form. In the following we use this matrix to show that the linear map arising in the formulation of Theorem 4 is injective.

Applying the transformation \mathbf{S} , to the linearised matrices of equation (59) —see also equation (63)— we get that

$$\mathbf{S}^{-1} \xi_j \mathring{\mathbf{A}}^j \mathbf{S} = \begin{pmatrix} 0 & 0 & \mathbf{A}_{5 \times 3}^{(1)} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{A}_{3 \times 3}^{(2)} & 0 & 0 & 0 \\ \mathbf{A}_{3 \times 5}^{(3)} & (\mathbf{A}^{(2)})_{3 \times 3}^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{A}_{3 \times 3}^{(4)} & 0 \\ 0 & 0 & 0 & (\mathbf{A}^{(4)})_{3 \times 3}^T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where

$$\mathbf{A}^{(1)} = \begin{pmatrix} \frac{2\omega}{3a} \frac{H\dot{\mathcal{V}}''}{2H\dot{\mathcal{V}}'' + \dot{\psi}\dot{\mathcal{V}}'} \xi_1 & \frac{2\omega}{3a} \frac{H\dot{\mathcal{V}}''}{2H\dot{\mathcal{V}}'' + \dot{\psi}\dot{\mathcal{V}}'} \xi_2 & \frac{2\omega}{3a} \frac{H\dot{\mathcal{V}}''}{2H\dot{\mathcal{V}}'' + \dot{\psi}\dot{\mathcal{V}}'} \xi_3 \\ \frac{\omega}{3a} \frac{\dot{\psi}\dot{\mathcal{V}}'}{2H\dot{\mathcal{V}}'' + \dot{\psi}\dot{\mathcal{V}}'} \xi_1 & \frac{\omega}{3a} \frac{\dot{\psi}\dot{\mathcal{V}}'}{2H\dot{\mathcal{V}}'' + \dot{\psi}\dot{\mathcal{V}}'} \xi_2 & \frac{\omega}{3a} \frac{\dot{\psi}\dot{\mathcal{V}}'}{2H\dot{\mathcal{V}}'' + \dot{\psi}\dot{\mathcal{V}}'} \xi_3 \\ 0 & 0 & 0 \\ 0 & -\frac{\omega}{a} \xi_2 & \frac{\omega}{a} \xi_3 \\ -\frac{\omega}{a} \xi_1 & 0 & \frac{\omega}{a} \xi_3 \end{pmatrix}, \quad \mathbf{A}^{(2)} = \begin{pmatrix} \frac{\omega}{a} \xi_2 & \frac{\omega}{a} \xi_1 & 0 \\ \frac{\omega}{a} \xi_3 & 0 & \frac{\omega}{a} \xi_1 \\ 0 & \frac{\omega}{a} \xi_3 & \frac{\omega}{a} \xi_2 \end{pmatrix},$$

$$\mathbf{A}^{(3)} = \begin{pmatrix} \frac{\omega}{a}\xi_1 & \frac{\omega}{a}\xi_1 & 0 & \frac{\omega}{3a}\xi_1 & -\frac{2\omega}{3a}\xi_1 \\ \frac{\omega}{a}\xi_2 & \frac{\omega}{a}\xi_2 & 0 & -\frac{2\omega}{3a}\xi_2 & \frac{\omega}{3a}\xi_2 \\ \frac{\omega}{a}\xi_3 & \frac{\omega}{a}\xi_3 & 0 & \frac{\omega}{3a}\xi_3 & \frac{\omega}{3a}\xi_3 \end{pmatrix}, \quad \mathbf{A}^{(4)} = \begin{pmatrix} 0 & \frac{\omega}{2a}\xi_1 & -\frac{\omega}{2a}\xi_2 \\ -\frac{\omega}{2a}\xi_1 & 0 & \frac{\omega}{2a}\xi_3 \\ \frac{\omega}{2a}\xi_2 & -\frac{\omega}{2a}\xi_3 & 0 \end{pmatrix}.$$

It is also noticed that the diagonality of $\mathring{\mathbf{A}}^0$ implies that it is invariant by \mathbf{S} . It follows that the linear map $i\mathring{\mathbf{A}}^0 + i\mathring{\mathbf{A}}(t, \xi)$ has determinant

$$\frac{i}{8a^{10}} (\omega^6 |\xi|^6 + 5a^2 \omega^4 |\xi|^4 + 8a^4 \omega^2 |\xi|^2 + 4a^6) (\omega^4 |\xi|^4 + 8a^2 \omega^2 |\xi|^2 + 16a^4),$$

where $|\xi|^2 \equiv \xi_1^2 + \xi_2^2 + \xi_3^2$. Therefore, for all ξ , a and ω the determinant is nonzero, and by Theorem 4 one finds that Assumption (i) of Definition 1 holds.

5.6 Verifying Assumption (iii) of Definition 1

In order to verify Assumption (iii) of the *relaxed stability eigenvalue condition*, we just need to consider the block matrix of the nonlinear matrix, $\check{\mathbf{B}}$, associated to the perturbation variables $(\check{\phi}, \check{\psi}, \check{\chi}_{11}, \check{\chi}_{22}, \check{\chi}_{33})$. A simple inspection of the linearisation process allow us to verify that this block matrix has zeros in all its entries. Therefore Assumption (iii) follows directly.

6 The main result

We summarise the discussion of previous sections in the following theorem:

Theorem 5. *Let ϕ be a smooth real scalar field with a selfinteracting potential $\mathcal{V}(\phi)$ in a flat expanding Friedmann-Robertson-Walker spacetime subject to the nonlinear evolution equations*

$$\frac{d\check{\phi}}{dt} = \check{\psi} \quad \text{and} \quad \frac{d\check{\psi}}{dt} = -3H\check{\psi} - \frac{d\check{\mathcal{V}}}{d\check{\phi}}.$$

Suppose there exists a constant H_0 such that $0 < H_0 \leq H(t)$, for all $t \geq 0$, and a potential satisfying conditions

$$\frac{\check{\mathcal{V}}'}{\check{\psi}} < -\frac{H}{2} \quad , \quad 2H \frac{\check{\mathcal{V}}'}{\check{\psi}} < -\frac{dH}{dt} - 2H^2 \quad , \quad \check{\mathcal{V}}'' > -\check{\psi} \frac{\check{\mathcal{V}}'}{2H} \quad \text{and} \quad \check{\mathcal{V}}'' - \check{\psi} \frac{\check{\mathcal{V}}'}{3H} > 10 \left(-H^2 + \frac{\check{\psi}^2}{3} \right).$$

Then the FRW-nonlinear scalar field solution is stable in the sense that, given any initial data for small nonlinear perturbations $\check{\mathbf{u}}_0$ whose norm $\|\check{\mathbf{u}}\|_{H^5(\mathbb{T}^3)}$ is finite, there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon \leq \epsilon_0$ a solution to the nonlinear perturbations exists for all times and decays exponentially to zero or converges to constant values.

Remark. Our main result can be understood as a *cosmic no-hair theorem* for a wide class of scalar field cosmologies. We note that our results can be useful when considering numerical simulations of these models as it gives conditions for exponential decay during the evolutions. It could, therefore, be used to test the consistency of the numerical steps.

6.1 Power law solutions with exponential potential

As a direct application of Theorem 5 we discuss the so-called *power-law* solutions to an exponential potential and show that this class of solutions is covered by our analysis.

Consider the reference solution for a scalar field ϕ with exponential potential

$$\mathcal{V}(\phi) = \Lambda e^{\lambda\phi},$$

with Λ, λ positive constants, given by the metric

$$ds^2 = -dt^2 + t^{2p} \delta_{ij} dx^i dx^j$$

with

$$\phi(t) = \frac{\sqrt{2p}}{2} \ln \left(\frac{p(3p-1)}{\Lambda t^2} \right), \quad \psi(t) = -\frac{\sqrt{2p}}{t}$$

with $p > 1/3$ and $\lambda^2 p = 2$. A direct computation shows that the conditions in Theorem 5 are satisfied if

$$p > \frac{\sqrt{61}-1}{15} \approx 0.454.$$

Thus, for this range of values one can conclude that *sufficiently small nonlinear perturbations to power-law solutions decay exponentially to zero or converge to constant values*.

Remark 1. It is well known that for $p > 1$ (i.e. $\lambda < \sqrt{2}$), the exact solution described in the previous lines has accelerated expansion — the so-called *power law inflation*. Our main result, Theorem 5 shows that one has decay of the nonlinear perturbations even in the absence of accelerated expansion if $p \in (\frac{\sqrt{61}-1}{15}, 1]$. This is, to the best of our knowledge, a new result —the possibility of having decay in the marginal case $p = 1$ has been discussed in [12, 13].

Remark 2. Simple numerical simulations of the evolution of linearised perturbations supports the results of our analysis. A detailed discussion of these is beyond the purposes of the present work.

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